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New Approximability Results for Two-Dimensional Bin Packing

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Abstract

We study the two-dimensional bin packing problem: Given a list of n rectangles the objective is to find a feasible, i.e. axis-parallel and non-overlapping, packing of all rectangles into the minimum number of unit sized squares, also called bins. Our problem consists of two versions; in the first version it is not allowed to rotate the rectangles while in the other it is allowed to rotate the rectangles by $90°$, i.e. to exchange the widths and the heights. Two-dimensional bin packing is a generalization of its one-dimensional counterpart and is therefore strongly $\mathcal{NP}$-hard. Furthermore Bansal et al. [2] showed that even an $\mathcal{APTAS}$ is ruled out for this problem, unless $\mathcal{P} = \mathcal{NP}$. This lower bound of asymptotic approximability was improved by Chlebík & Chlebíková [5] to values $1 + 1/3792$ and $1 + 1/2196$ for the version with and without rotations, respectively. On the positive side there is an asymptotic $1.69..$ approximation by Caprara [4] without rotations and an asymptotic $1.52...$ approximation by Bansal et al. [1] for both versions.

We give a new asymptotic upper bound for both versions of our problem: For any fixed ε and any instance that fits optimally into OPT bins, our algorithm computes a packing into $(3/2 + \varepsilon) \cdot \mathrm{OPT} + 69$ bins in the version without rotations and $(3/2 + \varepsilon) \cdot \mathrm{OPT} + 39$ bins in the version with rotations. The algorithm has polynomial running time in the input length.

In our new technique we consider an optimal packing of the rectangles into the bins. We cut a small vertical or horizontal strip out of each bin and move the intersecting rectangles into additional bins. This enables us to either round the widths of all wide rectangles, or the heights of all long rectangles in this bin. After this step we round the other unrounded side of these rectangles and we achieve a solution with a simple structure and only few types of rectangles. Our algorithm initially rounds the instance and computes a solution that nearly matches the modified optimal solution.

Keywords: Scheduling and Resource Allocation Problems, Bin Packing, Rectangle Packing, Approximation Algorithms

1 Introduction

In the two-dimensional bin packing problem it is desired to pack a list $I = \{r_1, \ldots, r_n\}$ of rectangles with heights h_i and widths w_i into the smallest possible number of unit sized squares,

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also called bins. The rectangles have to be packed axis-parallel and may not overlap. Our problem consists of two versions; in the first version it is not allowed to rotate the rectangles while in the other it is allowed to rotate the rectangles by $90°$, i.e. to exchange the widths and the heights. Two-dimensional packing problems have many real world applications that can be found in the area of scheduling, chip design and logistics. In particular the version of the two-dimensional bin packing problem with rotations can be used for example for stock-cutting, when we want to cut some items out of some sheets of raw material. The version without rotations is for example used for the print and web layout, when we want to place all ads into the minimum number of pages.

Related Work Two-dimensional bin packing is a generalization of its one-dimensional counterpart (where each rectangle has height 1) and is therefore strongly $\mathcal{NP}$-hard. Furthermore Bansal et al. [2] showed that even an $\mathcal{APTAS}$ is ruled out. This asymptotic lower bound was further improved by Chlebík & Chlebíková [5] to values $1 + 1/3792$ and $1 + 1/2196$ for the version with and without rotations, respectively. On the positive side there is an asymptotic 2.125-approximation by Chung et al. [6]. The $\mathcal{AFPTAS}$ of Kenyon & Rémila [19] and Jansen & van Stee [17] for the related strip packing problem can be used to achieve an asymptotic $2 + \varepsilon$-approximation for the two-dimensional bin packing problem without and with rotations, respectively. Caprara [4] gave the first asymptotic approximation algorithm for the version without rotations that breaks the barrier of 2. The asymptotic approximation ratio of this algorithm is arbitrary close to the harmonic number $T_\infty = 1.69...$ This result was further improved by Bansal et al. [1] with an asymptotic approximation ratio of arbitrary close to $\ln(T_\infty + 1) = 1.52..$ with and without rotations. The additive constant of this algorithm depends on a precision ε of this algorithm.

In the non-asymptotic setting without rotations there is a 3-approximation by Zhang [23] and by Harren & van Stee [12] with an improved running time. Harren & van Stee [12] also developed a non-asymptotic 2-approximation with rotations. Independently this approximation guarantee is also achieved for the version without rotations by Harren & van Stee [13] and Jansen et al. [15]. These results match the non-asymptotic lower bound of this problem, unless $\mathcal{P} = \mathcal{NP}$.

Our Contribution We present the following result for the two-dimensional bin packing problem with and without rotations:

Theorem 1. *For any $\varepsilon > 0$, there is an approximation algorithm A which produces a packing of a list I of n rectangles in $A(I)$ bins such that*

$$A(I) \leq (3/2 + \varepsilon) \cdot \mathrm{OPT}(I) + 69.$$

The running time of A is polynomial in n.

This result is an important step in closing the gap between the current asymptotic lower bound and the former best asymptotic approximation ratio. Furthermore, since we have a small additive constant of 69, our algorithm already computes better results for instances with $\mathrm{OPT}(I) \geq 200$ and $\varepsilon \leq 1/8$, or for $\mathrm{OPT}(I) \geq 150$ and $\varepsilon \leq 1/30$ than the non-asymptotic 2-approximations.

In the version that allows rotation we can further improve the additional constant to 39. We obtain the following result.

Theorem' 1. *For any $\varepsilon > 0$, there is an approximation algorithm A which produces a packing of a list I of n rectangles that are allowed to be rotated in $A(I)$ bins such that*

$$A(I) \leq (3/2 + \varepsilon) \cdot \mathrm{OPT}(I) + 39.$$

The running time of A is polynomial in n.

Techniques The main idea of our work is to analyse an arbitrary solution of the two-dimensional bin packing problem. Here it does not matter whether the rectangles are rotated or not. We cut in each bin a small vertical or horizontal strip out of the solution, i.e. we move some rectangles to additional bins, so that a horizontal or vertical strip at one side of the bin is completely free of rectangles. We prove that this is possible for any bin in any possible solution. At these modification steps, we do not rotate the rectangles in order to ensure that it also works for the version where rotations are not allowed. When we have removed a vertical strip of some width ε_c, it is possible to round the widths of all rectangles of width at least ε_c to a multiple of $\varepsilon_c^2/2$ and place them also on an x-coordinate whose value is a multiple of $\varepsilon_c^2/2$. When we have removed a horizontal strip of height ε_c we are able to round the heights of all rectangles of height at least ε_c to a multiple of $\varepsilon_c^2/2$. These rectangles are placed on a y-coordinate whose value is a multiple of $\varepsilon_c^2/2$. It follows that our modified solution consists of two different types of bins. The packing of the bins of the first type satisfy the following property.

Property 1. *The width and the x-coordinate of each rectangle in B_i of width at least ε_c is a multiple of $\varepsilon_c^2/2$.*

The packing of the bins of the second type satisfy the analogous property for rounding the heights:

Property 2. *The height and the y-coordinate of each rectangle in B_i of height at least ε_c is a multiple of $\varepsilon_c^2/2$.*

We ensure one of these properties also on the additional bins that are used to modify the solution and we obtain the following main result of our work:

Theorem 2. *For any value ε_c, with $1/\varepsilon_c$ being a multiple of 24, and for any solution that fits into m bins, we are able to round up the widths and the heights of the rectangles so that they fit into $(3/2 + 5\varepsilon_c) \cdot m + 37$ bins while the packing of each of the bins satisfies either Property 1 or Property 2.*

After having rounded one side of the rectangles the rounding technique for the unrounded side is fairly standard in the theory of packing algorithms. In general, we employ the rounding technique used in the $\mathcal{AFPTAS}$ by Kenyon & Rémila [19]. Small rectangles are packed in containers using some techniques by Jansen & Solis-Oba [16]. Furthermore, we use the algorithm of Steinberg [21], to pack some medium rectangles.

In the non-rotational version, our algorithm initially uses a flow network to assign some big rectangles that have both side lengths at least ε_c to bins of the first and second type. The same flow network is used in the setting that allows rotation to rotate these rectangles. The remaining small, (rotated) long and (rotated) wide rectangles are packed into containers with a modified version of the algorithm by Kenyon & Rémila [19]. Afterwards we pack the containers and the big rectangles with an integer linear program into the bins. There are only minor differences to improve the additional constant in the version that allows rotation. We state these differences at the end of each section.

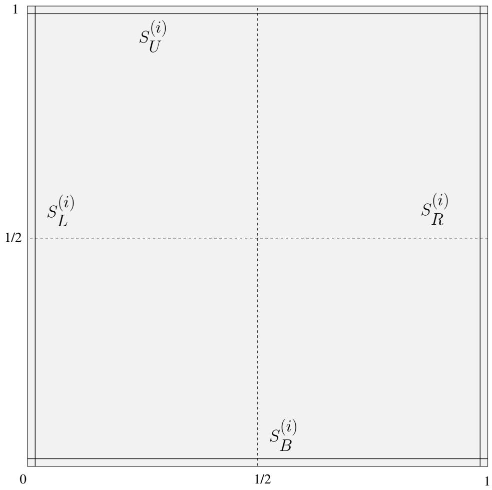


Figure 1: Definition of $S_U^{(i)}, S_B^{(i)}, S_L^{(i)}$ and $S_R^{(i)}$

2 Modifying a Packing

In the following sections, we consider an arbitrary solution, which does not have to be the optimal one, of the rectangles in m bins. We set a coordinate system to each bin, with the origin $(0, 0)$ in the lower left corner and with the coordinate $(1, 1)$ in the upper right corner. The lower left corner of the rectangle r_j is placed at the position (x_j, y_j) and the upper right corner at the position (x'_j, y'_j). The area of r_j is defined by $a_j := h_j \cdot w_j$. For a set X of rectangles, we have $h(X) := \sum_{r_j \in X} h_j$, $w(X) := \sum_{r_j \in X} w_j$ and $a(X) := \sum_{r_j \in X} h_j \cdot w_j$ for the total height, total width and total area of the rectangles in X. The maximal occurring width and height in X is defined by $w_{\max}(X) := \max_{r_j \in X} w_j$ and $h_{\max} := \max_{r_j \in X} h_j$.

Sometimes, we define a certain rectangular or Γ-shaped region in a bin B_i of our solution. These regions are defined by a closed traverse starting at the lower left corner. A rectangular region, defined by some corner points $(x_1, y_1), (x_2, y_1), (x_2, y_2)$ and (x_1, y_2), is also defined by the Cartesian product $[x_1, x_2] \times [y_1, y_2]$.

Let $\varepsilon_c < 1$ be a value, so that $1/\varepsilon_c$ is a multiple of 24. In order to round the rectangles in our solution, we cut a horizontal strip of height ε_c and width 1 or a vertical strip of width ε_c and height 1 out of each bin B_i. Therefore, we clear always one of the four strips at the sides of the bin, i.e. we remove all rectangles that intersect one of them (except the bins B_i with a very large rectangle that intersects simultaneously all four strips).

Denote the strips of width 1 and height ε_c at the top and at the bottom of the strip by $S_U^{(i)} := [0, 1] \times [1 - \varepsilon_c, 1]$ and $S_B^{(i)} := [0, 1] \times [0, \varepsilon_c]$. The strips of height 1 and width ε_c to the right and left of the bin are called $S_R^{(i)} := [1 - \varepsilon_c, 1] \times [0, 1]$ and $S_L^{(i)} := [0, \varepsilon_c] \times [0, 1]$ (cf. Figure 1). There are two kinds of rectangles that intersect these strips. The set of rectangles that lies completely in one of these strips $S_K^{(i)}$, $K \in \{U, B, R, L\}$ is denoted by $C_K^{(i)}$; the set of rectangles that does not lie completely inside a strip but intersects this strip is denoted by $I_K^{(i)}$.

In the following, we want to prove that the union of all sets $C_R^{(i)}, C_L^{(i)}, C_U^{(i)}$ and $C_B^{(i)}$, $i \in \{1, \ldots, m\}$, covers a very small total area and can be moved into few additional bins.

Lemma 1. *We move all rectangles in $C_R^{(i)}, C_L^{(i)}, C_U^{(i)}$ and $C_B^{(i)}$ for all $i \in \{1, \ldots, m\}$ into $4\varepsilon_c m + 2$ additional bins. The packing of these bins satisfy either Property 1 or Property 2.*

Proof. The rectangles in $C_R^{(i)}$ and $C_L^{(i)}$ are already packed into a strip of height 1 and width ε_c. We pack $1/\varepsilon_c$ of these strips into an additional bin. We have in total $2m$ strips to pack into extra bins. Hence, we need at most $\lceil 2\varepsilon_c \cdot m \rceil \leq 2\varepsilon_c m + 1$ bins. The rectangles of the width ε_c are placed on an x-coordinate whose value is a multiple of ε_c. This value is also a multiple of $\varepsilon_c^2/2$. The remaining rectangles have a width of less than ε_c and hence, this packing satisfies Property 1. The analogous packings for the rectangles in the strips S_U and S_R satisfy Property 2, and we need in total $4\varepsilon_c m + 2$ bins. □

In the following, we suppose that there is no rectangle completely situated in one of the strips $S_K^{(i)}$, $K \in \{U, B, R, L\}$ and $i \in \{1, \ldots, m\}$, but only rectangles that intersect them. Furthermore, we suppose that the rectangles that intersect $S_K^{(i)}$, i.e. the rectangles of $I_K^{(i)}$, touch the corresponding side of the bin. Therefore, we extend the widths or heights if necessary. Note that this is only for the ease of explanation, the rectangles are rounded later. The rectangles in the corners that intersect a vertical and a horizontal strip are extended in both directions so that they are placed directly in the corners. If there is no such rectangle in one corner, we employ a dummy rectangle of width and height ε_c. We denote the rectangles in the corners by $r_{u\ell}^{(i)} \in I_U^{(i)} \cap I_L^{(i)}; r_{ur}^{(i)} \in I_U^{(i)} \cap I_R^{(i)}; r_{b\ell}^{(i)} \in I_B^{(i)} \cap I_L^{(i)}$ and $r_{br}^{(i)} \in I_B^{(i)} \cap I_R^{(i)}$. Dummy rectangles of the width or the height ε_c are also filled in the remaining gaps. Hence, we suppose that $h(I_L^{(i)}) = 1, h(I_R^{(i)}) = 1, w(I_U^{(i)}) = 1$ and $w(I_B^{(i)}) = 1$. Consequently, the bins that contain a very large rectangle that simultaneously intersects all four strips, contain no further rectangles. The width and the height of this rectangle is rounded up to 1, and so the packing of these bins satisfies Property 1 and Property 2.

Lemma 2. *If B_i is a bin in our solution that contains a rectangle that simultaneously intersects the strips $S_U^{(i)}, S_B^{(i)}, S_R^{(i)}$ and $S_L^{(i)}$, then we are able to round up this rectangle and the packing satisfies Property 1.*

The rectangles intersecting at least one of these strips and having a height or a width larger than $1/2$ play a crucial part in our analysis. Consequently, let $L_U^{(i)} \subseteq I_U^{(i)}$ be the set of rectangles intersecting $S_U^{(i)}$ and having a height larger than $1/2$. $L_B^{(i)} \subseteq I_B^{(i)}$ is the set of rectangles intersecting $S_B^{(i)}$ and having a height larger than $1/2$. Furthermore, let $W_R^{(i)} \subseteq I_R^{(i)}$ and $W_L^{(i)} \subseteq I_L^{(i)}$ be the rectangles of a width larger than $1/2$ and intersecting $S_R^{(i)}$ or $S_L^{(i)}$, respectively. The rectangle of a maximum height in $L_U^{(i)}$ is denoted by $r_u^{(i)}$ and that of $L_B^{(i)}$ is denoted by $r_b^{(i)}$. The rectangle of a maximum width in $W_L^{(i)}$ is denoted by $r_\ell^{(i)}$ and that of $W_R^{(i)}$ is denoted by $r_r^{(i)}$ (cf. Figure 2(a)).

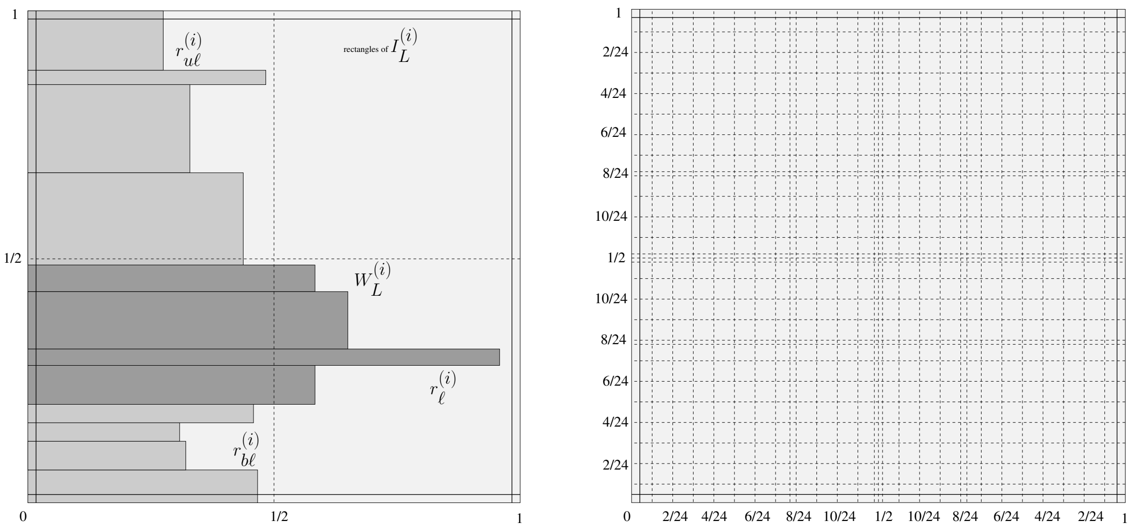
We separate each bin B_i into 28 horizontal and vertical strips. Therefore, let

$$\mathrm{IN} = \bigcup_{i=0}^{11} \{i/24\} \cup \{8/24 - \varepsilon_c, 12/24 - \varepsilon_c\},$$

be a set of numbers and let

$$\mathrm{IN}' := \mathrm{IN} \cup \{12/24\}$$

be the extended set. We assume that these sets are sorted according to non-decreasing values. We use two consecutive numbers in_i and in_{i+1} of IN' as x-coordinates of each vertical strip of the height 1(cf. Figure 2(b)). Therefore, we define the 14 vertical strips on the left and right side of the bin by $\mathrm{VL}_{in_i}^{(i)} := [in_i, in_{i+1}] \times [0, 1]$ and by $\mathrm{VR}_{in_i}^{(i)} := [1 - in_{i+1}, 1 - in_i] \times [0, 1]$. Analogously, we define the 14 horizontal strips of the width 1 in the lower half and upper half of B_i by $\mathrm{HB}_{in_i}^{(i)} := [0, 1] \times [in_i, in_{i+1}]$ and $\mathrm{HU}_{in_i}^{(i)} := [0, 1] \times [1 - in_{i+1}, 1 - in_i]$.



(a) Definition of the rectangles intersecting $S_L^{(i)}$ (b) The horizontal and vertical strips

Figure 2: Definition of rectangles and strips

If we completely remove one vertical strip of the width ε_c in one bin, including all rectangles that intersect it, we are able to round up the widths of all rectangles that have a width of at least ε_c.

Lemma 3. *If there is a vertical strip of the width ε_c free of rectangles in a bin B_i, then we are able to round up the widths of the rectangles so that the packing of B_i satisfies Property 1.*

Proof. W.l.o.g. we assume that $S_R^{(i)}$ is the free strip of rectangles, since we can move all rectangles on the right of one free strip by ε_c to the left. We divide the remaining bin into $2/\varepsilon_c - 2$ vertical strips of the width $\varepsilon_c/2$ by introducing vertical lines at the position $i \cdot \varepsilon_c/2$, for $i \in \{1, \ldots, 2/\varepsilon_c - 2\}$. Each rectangle of a width larger than ε_c intersects at least 3 of these strips and hence crosses at least 2 vertical lines. In a next step we enlarge the strips to a width of $\varepsilon_c/2 + \varepsilon_c^2/2$ by giving some extra space to a rectangle each time it intersects one of these vertical lines by $\varepsilon_c^2/2$. The total width of all strips is $(2/\varepsilon_c - 2) \cdot (\varepsilon_c/2 + \varepsilon_c^2/2) = (1/\varepsilon_c - 1) \cdot (\varepsilon_c + \varepsilon_c^2) = 1 + \varepsilon_c - \varepsilon_c - \varepsilon_c^2 = 1 - \varepsilon_c^2 \leq 1$.

Let r_k be a rectangle that intersects at least 2 vertical lines and that has a width of $w_k \in (i\varepsilon_c^2/2, (i+1)\varepsilon_c^2/2]$ and an x-coordinate $x_k \in (j\varepsilon_c^2/2, (j+1)\varepsilon_c^2/2]$, for some values $i \in \{2/\varepsilon_c, \ldots, 2/\varepsilon_c^2 - 2/\varepsilon_c - 1\}$ and $j \in \{1, \ldots, 2/\varepsilon_c^2 - 2/\varepsilon_c - 1\}$. The extra space of r_k is at least $2 \cdot \varepsilon_c^2/2 = \varepsilon_c^2$ and is large enough for increasing the width from w_k to $\overline{w_k} := (i+1)\varepsilon_c^2/2$ and the x-coordinate from x_k to $\overline{x_k} := (j+1)\varepsilon_c^2/2$. All rectangles of a width larger than ε_c intersect at least 2 vertical lines and thus we round up their widths. It is possible that a rectangle r_k of a width exactly $w_k = \varepsilon_c$ does not intersect 2 vertical lines, because the x-coordinate is already a multiple of ε_c^2. In this case we do not have to change the position and the width of r_k. □

Analogously, we can prove the same result when there is a horizontal strip of the height ε_c free of rectangles.

Lemma 4. *If there is a horizontal strip of the height ε_c free of rectangles in a bin B_i, then we are able to round up the heights of the rectangles so that the packing of B_i satisfies Property 2.*

Packing with Rotations In the version with rotations, we always clear one of the vertical strips $S_L^{(i)}$ or $S_R^{(i)}$ of the width ε_c. This is possible since we are able to rotate the packing by $90°$. This also enables us to use less bins than in the version in which rotations are not allowed. Nevertheless, in the version with rotations we use the same techniques as in the version without rotations. Therefore, we first state the methods without rotations and explain afterwards the minor differences occurring, when we are allowed to rotate the rectangles. In Lemma 1, we obtain already an improvement of 1 additional bin, since we are able to rotate the rectangles in the horizontal strips. Thus, we have $4\varepsilon_c \cdot m$ vertical strips to pack into additional bins. The total number of additional bins is therefore $\lceil 4\varepsilon_c \cdot m \rceil \leq 4\varepsilon_c \cdot m + 1$. We obtain the following lemma.

Lemma' 1. *We move all rectangles in $C_R^{(i)}, C_L^{(i)}, C_U^{(i)}$ and $C_B^{(i)}$ for all $i \in \{1, \ldots, m\}$ into $4\varepsilon_c m + 1$ additional bins. The packing of these bins satisfies Property 1.*

2.1 Classify the Bins

In the Section 2.1 and Section 2.2 we explain how to clear a vertical or horizontal strip in each bin of our solution. We start by displaying the following lemma that has some impact on the structure of the packing in the remaining bins.

Lemma 5. *Let there be two bins B_1, B_2 in our solution, with vertical strips $S_{C_1}^{(1)}, S_{C_2}^{(2)}$ for $C_1 \in \{L, R\}$ and $C_2 \in \{L, R\}$. Furthermore, let there be an $x \in [0, 1/2]$, being a multiple of ε_c, and a value $y \in [0, 1/2]$. If the following conditions hold*

1.1. all rectangles of $W_{C_1}^{(1)}, W_{C_2}^{(2)}$ have a width of at most $1 - x$,

1.2. $h(W_{C_1}^{(1)}) \leq y$ and $h(W_{C_2}^{(2)}) \leq y$,

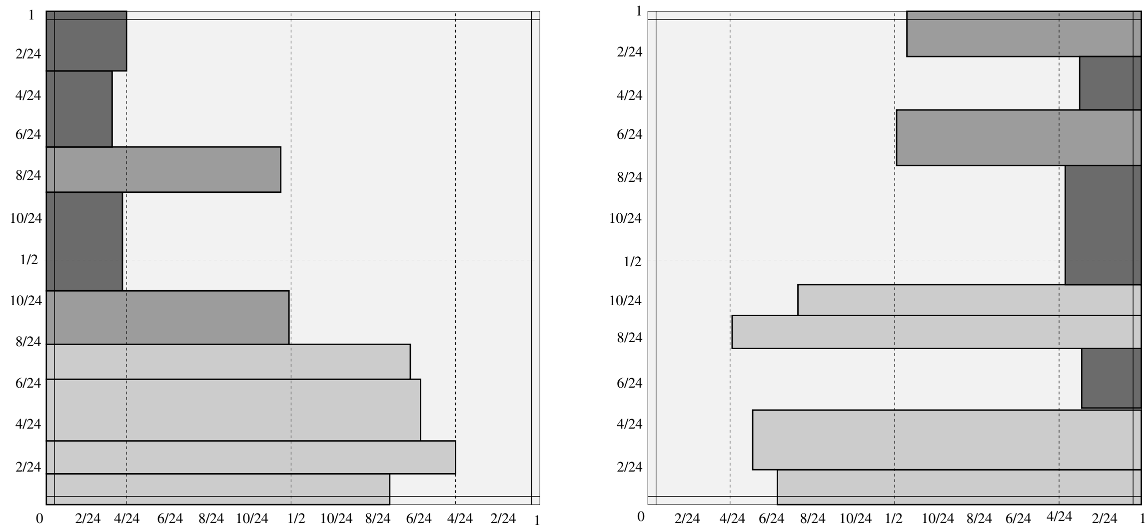
1.3. there are rectangles in the set $I_{C_1}^{(1)}$ and $I_{C_1}^{(2)}$ that have a width of at most x and a total height of at least y,

then we are able to round up the rectangles and rearrange them into three bins, while the packing of each of the bins satisfies Property 1.

Proof. We clear the strips $S_{C_1}^{(1)}$ and $S_{C_2}^{(2)}$ in the bins B_1 and B_2 and pack the intersecting sets of rectangles $I_{C_1}^{(1)}$ and $I_{C_2}^{(2)}$ into a new bin B_3 (cf. Figure 3). The rectangles of $I_{C_1}^{(1)}$ and $I_{C_2}^{(2)}$ each have a total height of 1 (including the dummy rectangles).

In a first step, we sort these rectangles according to their widths. The rectangles of $I_{C_1}^{(1)}$ are sorted according to non-increasing widths and placed with their x-coordinates at the position 0 in bin B_3. The rectangle with the maximal width is placed at the bottom of the bin and the rectangle with the minimum width is placed at the top. The rectangles of $I_{C_2}^{(2)}$ are sorted according to non-decreasing widths and are placed left aligned with their x'-coordinates at the position 1. Here, the rectangle with the minimum width is at the bottom of the bin and the rectangle with the maximum width is at the top (cf. Figure 4). To prove that these two columns of rectangles do not intersect, we look at the three regions between the horizontal lines at height $0, y, 1 - y$ and 1. Since $y \leq 1/2$ we have always $y \leq 1 - y$.

All rectangles in $W_{C_1}^{(1)}$ have a total height of at most y (cf. Condition 1.2), and are therefore placed in the left column below the horizontal line at height y. They have a width of at most $1 - x$ (cf. Condition 1.1). There are rectangles of a total height of at least y that have widths of at most x in $I_{C_1}^{(1)}$ (cf. Condition 1.3). These rectangles are placed in the right column below the horizontal line at height y. Consequently, the rectangles below the horizontal line at height



(a) Rectangles of $S_L^{(1)}$ in bin B_1

(b) Rectangles of $S_R^{(2)}$ in bin B_2

Figure 3: Using Lemma 5 with $y = 8/24$ and $x = 4/24$

y do not intersect each other. Vice versa, this also holds for the packing above the horizontal line at height $1 - y$. The rectangles that are placed between the horizontal lines at height y and $1 - y$ have widths of at most $1/2$. Thus, the rectangles in the two columns do not intersect each other.

After that there is a vertical strip of the width ε_c completely free of rectangles in bin B_1 and B_2. Hence, we are able to round the rectangles according to Lemma 3 in order to satisfy Property 1. The widths of the rectangles in bin B_3 are also rounded to the next largest multiple of $\varepsilon_c^2/2$, to values of at most $x, 1/2$ and $1 - x$, respectively. These values are all multiples of ε_c and therefore also multiples of $\varepsilon_c^2/2$ (for $1/\varepsilon_c = i \cdot 24$ and $x = j\varepsilon_c$ we have $x = j\varepsilon_c = 2j\varepsilon_c^2/(2\varepsilon_c) = (2 \cdot j \cdot i \cdot 24)\varepsilon_c^2/2 = (48 \cdot j \cdot i)\varepsilon_c^2/2$; furthermore, since $x = j\varepsilon_c \leq 1$ it is $1 - x = 1 - j\varepsilon_c = (1/\varepsilon_c - j)\varepsilon_c = (i \cdot 24 - j) \cdot \varepsilon_c = (48 \cdot (i \cdot 24 - j) \cdot i)\varepsilon_c^2/2)$.

□

The analogous lemma for rounding the heights is as follows. We omit the proof, since it is analogous to the proof of Lemma 5.

Lemma 6. *Let there be two bins B_1, B_2 in our solution, with horizontal strips $S_{C_1}^{(1)}, S_{C_2}^{(2)}$ for $C_1 \in \{U, B\}$ and $C_2 \in \{U, B\}$. Furthermore, let there be an $x \in [0, 1/2]$, being a multiple of ε_c, and a value $y \in [0, 1/2]$. If the following conditions hold*

1.4. all rectangles of $L_{C_1}^{(1)}, L_{C_2}^{(2)}$ have height at most $1 - x$,

1.5. $w(L_{C_1}^{(1)}) \leq y$ and $w(L_{C_2}^{(2)}) \leq y$,

1.6. there are rectangles in the set $I_{C_1}^{(1)}$ and $I_{C_1}^{(2)}$ that have a height of at most x and a total width of at least y,

then we are able to round up the rectangles and rearrange them into three bins, while the packing of each of the bins satisfies Property 2.

We are not able to use this lemma to all values of x and y, since there might be an unbounded number of them. Thus, we use a discretization and employ these lemmas only for all $x \in \mathrm{IN}$ and for $y \in \{0, 1/2\}$

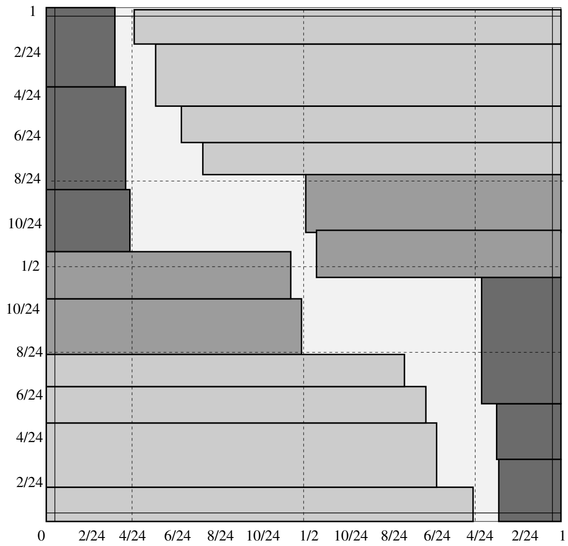


Figure 4: Combining the rectangles of $S_L^{(1)}$ and $S_R^{(2)}$

Lemma 7. *Let k denote the number of bins B_i in our solution, for which an $x \in \mathrm{IN}$ and a $y \in \{0, 1/2\}$ exists so that one of the following conditions holds:*

1.7. The total height of $W_L^{(i)}$ is at most y, all rectangles of $W_L^{(i)}$ have a width of at most $1-x$ and there are rectangles of a total height of at least y in $I_L^{(i)}$ that have a width of at most x.

1.8. The total height of $W_R^{(i)}$ is at most y, all rectangles of $W_R^{(i)}$ have a width of at most $1-x$ and there are rectangles of a total height of at least y in $I_R^{(i)}$ that have a width of at most x.

1.9. The total width of $L_B^{(i)}$ is at most y, all rectangles of $L_B^{(i)}$ have a height of at most $1-x$ and there are rectangles of a total width of at least y in $I_B^{(i)}$ that have a height of at most x.

1.10. The total width of $L_U^{(i)}$ is at most y, all rectangles of $L_U^{(i)}$ have a height of at most $1-x$ and there are rectangles of a total width of at least y in $I_U^{(i)}$ that have a height of at most x.

We are able to round the rectangles of these k bins and rearrange them into $3/2k + 15$ bins, while the packing of each of the bins satisfies either Property 1 or Property 2.

Proof. We separate these k bins into 30 sets. For each $x \in \mathrm{IN}$ we denote the set of bins, for which either Condition 1.7 or Condition 1.8 holds with $y = 1/2$, by $V_{x,1/2}$. Analogously, we denote the set of the remaining bins, for which either Condition 1.9 or Condition 1.10 holds with $y = 1/2$, by $H_{x,1/2}$. Furthermore, let V_0 denote the set of remaining bins, for which $W_L = \emptyset$ or $W_R = \emptyset$ holds and let H_0 denote the set of remaining bins, for which $L_B = \emptyset$ or $L_U = \emptyset$ holds. These are the bins that satisfy one of the four conditions with $y = 0$.

We employ Lemma 5 with each sequence of two bins in each set $V_{x,1/2}$ and V_0 and Lemma 6 with each sequence of two bins in each set $H_{x,1/2}$ and H_0. We need one additional bin for each set with an odd cardinality ℓ. This results in a packing of $3/2(\ell - 1) + 2$ bins. Consequently, we have at most $3/2(k - 30) + 2 \cdot 30 = 3/2k + 15$ bins in total, when all 30 sets have an odd cardinality. $\square$

In the following, we present some corollaries following from the lemma above. We prove that the packings in the remaining bins have a certain structure.

Corollary 1. *Let B_i be a bin in our solution for which Lemma 2 and Lemma 7 are not applicable. It follows that the sets $L_U^{(i)}, L_B^{(i)}, W_L^{(i)}$ and $W_R^{(i)}$ are non-empty and disjoint.*

Proof. Suppose by contradiction that $W_L^{(i)} = \emptyset$. It follows that all rectangles intersecting $S_L^{(i)}$ have a width of at most $1/2$. Hence, we have fulfilled Condition 1.7, with $y = 0$, which is a contradiction. The proof for $L_U^{(i)}, L_B^{(i)}$ and $W_R^{(i)}$ is analogous. Thus, the sets $L_U^{(i)}, L_B^{(i)}, W_L^{(i)}$ and $W_R^{(i)}$ are non-empty and as a consequence of Lemma 2 there is no rectangle simultaneous in all four sets.

Suppose by contradiction that there is a rectangle $r_1 \in L_U^{(i)} \cap L_B^{(i)}$. The rectangle r_1 has a height of 1. If $r_1 \in W_L^{(i)}$, then $r_1 \notin W_R^{(i)}$ and its x'-coordinate has to be larger than $1/2$. It follows that each rectangle in $W_R^{(i)}$ intersects r_1, which is a contradiction. If $r_1 \notin W_L^{(i)}$, then its x-coordinate has to be larger than $1/2$, since otherwise each rectangle in $W_L^{(i)}$ intersects r_1. However, each rectangle in $W_R^{(i)}$ intersects r_1, which is again a contradiction. Consequently, there is no rectangle in $L_U^{(i)} \cap L_B^{(i)}$ and analogously there is no rectangle in $W_L^{(i)} \cap W_R^{(i)}$.

Suppose by contradiction that there is a rectangle $r_1 \in L_U^{(i)} \cap W_L^{(i)}$. This rectangle has a width and a height of larger than $1/2$. Thus, its y-coordinate is less than $1/2$ and its x'-coordinate is larger than $1/2$. Each rectangle $r_2 \in W_R^{(i)}$ is therefore positioned below r_1 with a y'-coordinate less than $1/2$. If this was the case each rectangle in $L_B^{(i)}$ would intersect either r_1 or r_2, which is a contradiction. The proof for the disjunction of the remaining sets is analogously. □

This result enables us to do a first analysis of the packings in the remaining bins. Let B_i be a bin, in which Lemma 2 and Lemma 7 are not applicable. Suppose by contradiction that there are two (not necessarily distinct) rectangles $r_1, r_2 \in L_U^{(i)}$ so that r_1 has the x-coordinate $x_1 \leq 1/2$ and r_2 has the x'-coordinate $x'_2 \geq 1/2$. The rectangles $r_3 \in W_L^{(i)}$ and $r_4 \in W_R^{(i)}$ have to lie below r_1 and r_2 and their y'-coordinates are less than $1/2$. Hence, each rectangle in $L_B^{(i)}$ intersects either r_3 or r_4, which is a contradiction.

Consequently, all x- and x'-coordinates of the rectangles in $L_U^{(i)}$ are either less than $1/2$ or larger than $1/2$. W.l.o.g. we assume that all x-coordinates are less than $1/2$, since we are able to mirror the packing at the vertical line at the x-coordinate $1/2$. All rectangles of $W_L^{(i)}$ are placed below the rectangles of $L_U^{(i)}$ and their y'-coordinates are less than $1/2$. Consequently, the rectangles of $L_B^{(i)}$ are on the right of the rectangles in $W_L^{(i)}$ and their x-coordinates are larger than $1/2$. The rectangles of $W_R^{(i)}$ are situated above the rectangles in $L_B^{(i)}$ and their y-coordinates are larger than $1/2$ (cf. Figure 5). We obtain the following structural theorem.

Theorem 2. *Consider a bin B_i in our solution, for which Lemma 2 and Lemma 7 are not applicable. The rectangles of $L_U^{(i)}$ are, w.l.o.g., completely in the left half of these bins (all x'-coordinates are less than $1/2$); the rectangles of $L_B^{(i)}$ are completely in the right half of these bins (all x-coordinates are larger than $1/2$); the rectangles of $W_L^{(i)}$ are completely in the lower half of these bins (all y'-coordinates are less than $1/2$) and the rectangles of $W_R^{(i)}$ are completely in the upper half of these bins (all y-coordinates are larger than $1/2$).*

It is very useful that this structure remains the same when turning the bin by $90°$, $180°$ and $270°$ since sometimes we use analogous arguments. Note that by turning the bin by $180°$, we rotate the packing but not the rectangles.

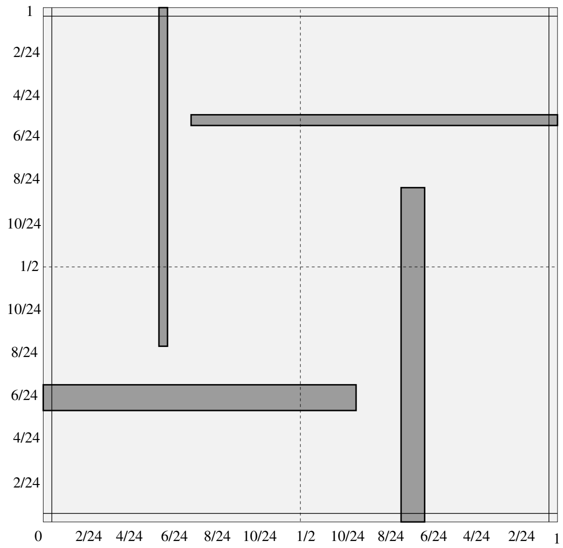
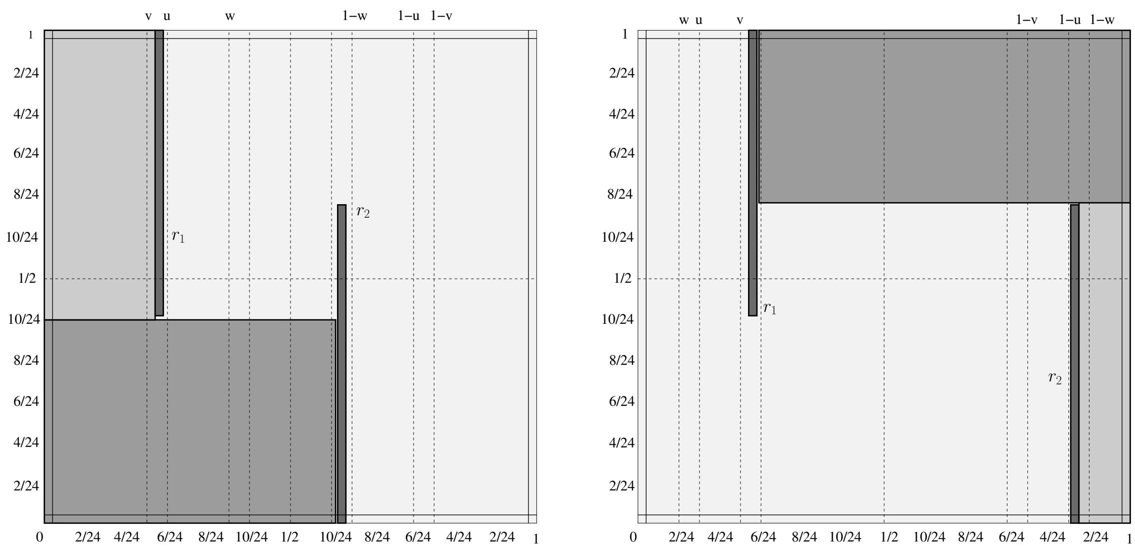


Figure 5: A packing as described in Theorem 2

Corollary 2. *Let B_i be a bin in our solution for which Lemma 2 and Lemma 7 are not applicable. Furthermore, let there be a rectangle r_1 in $L_U^{(i)}$ with an x'-coordinate in an interval $\mathrm{VL}_v^{(i)}$, for a $v \in \mathrm{IN}$. It follows that the x-coordinates of all rectangles in $L_B^{(i)}$ are situated in $\mathrm{VR}_v^{(i)}$.*

Proof. Let w be an element of IN. Suppose by contradiction that $w \neq v$ and that there is a rectangle r_2 in $L_B^{(i)}$ with an x-coordinate in $\mathrm{VR}_w^{(i)}$ (cf. Figure 6).



(a) Case 1 with $v = 5/24$ and $w = 9/24$

(b) Case 2 with $v = 5/24$ and $w = 2/24$

Figure 6: The two cases of Corollary 2

Case 1, $w > v$. Let $u \leq w$ be the successor of v in IN, i.e. $\mathrm{VL}_v^{(i)} = [v, u] \times [0, 1]$. The widths of all rectangles in $I_L^{(i)}$ that lie above the horizontal line at the y-coordinate y_1 or that intersect with it are bounded by the rectangle r_1. Hence, their widths are bounded by the value $u \leq w$. The height of r_1 is larger than $1/2$, hence $y_1 < 1/2$. Consequently, there are rectangles of the total height of at least $1/2$ that have a width of at most w. The remaining rectangles in $I_L^{(i)}$ that lie below the horizontal line at the y-coordinate y_1 are bounded by the rectangle r_2. The x-coordinate of r_2 is within $\mathrm{VL}_w^{(i)}$, it follows that these rectangles have a width of at most $1-w$. We have fulfilled Condition 1.7 with $y = 1/2$ and $x = w$, which is a contradiction.

Case 2, $w < v$. Let $u \leq v$ be the successor of w in IN', i.e. $\mathrm{VR}_w^{(i)} = [1-u, 1-w] \times [0,1]$. We use the same argumentation as in the first case on the strip $S_R^{(i)}$. The widths of the rectangles in $I_R^{(i)}$ that are positioned below the y'-coordinate y_2' are bounded by the rectangle r_2. Their total height is larger than $1/2$ and their widths are at most $u \leq v$. The widths of the rectangles in $I_R^{(i)}$ that lie above the horizontal line at the y'-coordinate y_2' are bounded by r_1. The x'-coordinate of r_1 is in $\mathrm{VL}_v^{(i)}$ and hence the widths are bounded by $1-v$. It follows that we satisfy Condition 1.8 with $y = 1/2$ and $x = v$, which is a contradiction. □

Consequently, if there is a rectangle in $L_U^{(i)}$ with its x'-coordinate in an interval $\mathrm{VL}_v^{(i)}$, all rectangles of $L_B^{(i)}$ have their x-coordinates in the interval $\mathrm{VR}_v^{(i)}$. Furthermore, we can use this corollary after turning the bin by $180°$. We obtain: if there is a rectangle in $L_B^{(i)}$ with its x-coordinate in an interval $\mathrm{VR}_v^{(i)}$, then all rectangles of $L_U^{(i)}$ have their x'-coordinates in the interval $\mathrm{VL}_v^{(i)}$. Hence, the x'-coordinates of all rectangles in $L_U^{(i)}$ are in $\mathrm{VL}_v^{(i)}$ and the x-coordinates of all rectangles in $L_B^{(i)}$ are in $\mathrm{VR}_v^{(i)}$. The same holds for the wide rectangles intersecting $S_R^{(i)}$ and $S_L^{(i)}$ by employing this corollary on the bin turned by $90°$. Thus, the y'-coordinates of all rectangles in $W_L^{(i)}$ are in $\mathrm{HB}_h^{(i)}$ and the y-coordinates of all rectangles in $W_R^{(i)}$ are in $\mathrm{HU}_h^{(i)}$, for some $h \in \mathrm{IN}$.

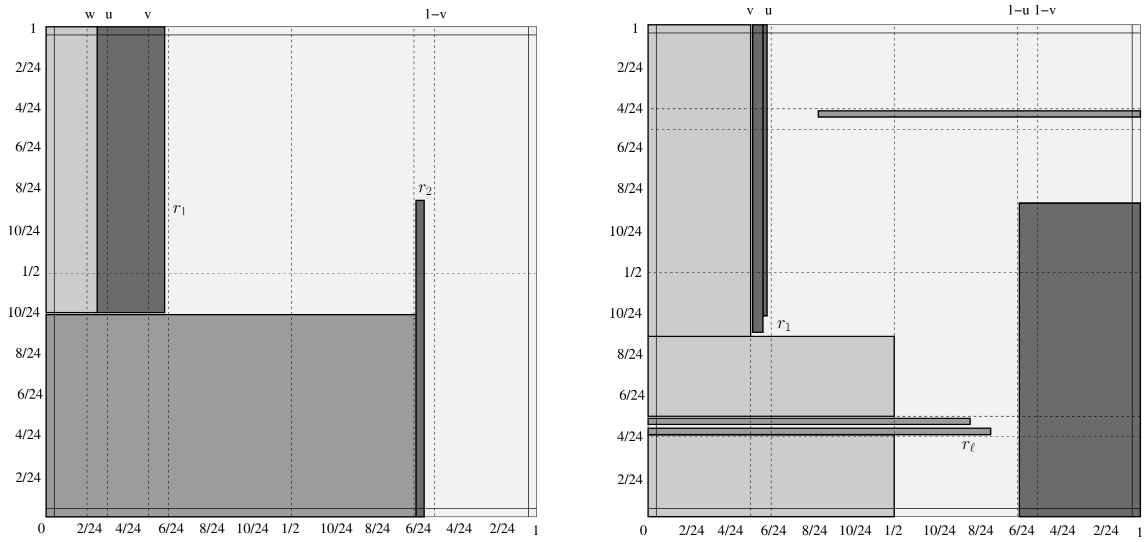
Corollary 3. *Let B_i be a bin in our solution for which Lemma 2 and Lemma 7 are not applicable. Furthermore, let there be a rectangle $r_1 \neq r_{u\ell}^{(i)}$ in $L_U^{(i)}$ with x'-coordinate in an interval $\mathrm{VL}_v^{(i)}$ for $v \in \mathrm{IN}$ ($r_{u\ell}^{(i)}$ is the rectangle in the upper-left corner). It follows that the x-coordinate of r_1 is also situated in $\mathrm{VL}_v^{(i)}$.*

Proof. Suppose by contradiction that there is a member $w < v$ of IN and the x-coordinate of r_1 is in $\mathrm{VL}_w^{(i)}$. It holds that $r_1 \neq r_{u\ell}^{(i)}$ and hence r_1 does not intersect $S_L^{(i)}$ (cf. Figure 7(a)).

Let $u \leq v$ be the successor of w in IN', i.e. $\mathrm{VL}_w^{(i)} = [w, u] \times [0, 1]$. The widths of the rectangles in $I_L^{(i)}$ that lie above the horizontal line at y-coordinate y_1 or that intersect this line are bounded by r_1. It follows that their total height is at least $1 - y_1 > 1/2$ and their widths are at most $u \leq v$. Furthermore, as a consequence of Corollary 1 and Corollary 2 there exists a rectangle r_2 of $L_B^{(i)}$ that has its x-coordinate x_2 within $\mathrm{VR}_v^{(i)}$. Thus, all remaining rectangles in $I_L^{(i)}$ that are below the horizontal line at y-coordinate y_1 have a bounded width of at most $1-v$. Consequently, we satisfy Condition 1.7 with $y = 1/2$ and $x = v$, which is a contradiction. □

As a consequence of Corollary 2 and Corollary 3, all rectangles of $L_U^{(i)}$ except $r_{u\ell}^{(i)}$, if $r_{u\ell}^{(i)} \in L_U^{(i)}$, are completely situated in an interval $\mathrm{VL}_v^{(i)}$. Again, we employ this corollary on each side of the bin and as a consequence, we achieve that all rectangles of $L_B^{(i)} \setminus \{r_{br}^{(i)}\}$ are completely in $\mathrm{VR}_v^{(i)}$. All rectangles of $W_L^{(i)} \setminus \{r_{br}^{(i)}\}$ are completely in an interval $\mathrm{HB}_h^{(i)}$ and all rectangles of $W_R^{(i)} \setminus \{r_{ur}^{(i)}\}$ are completely in $\mathrm{HU}_h^{(i)}$.

Corollary 4. *Let B_i be a bin in our solution, for which Lemma 2 and Lemma 7 are not applicable and let there be a rectangle $r_1 \in L_U^{(i)}$ with x'-coordinate in an interval $\mathrm{VL}_v^{(i)}$. It follows that*



(a) Corollary 3; $v = 5/24$, $w = 2/24$.

(b) Corollary 4; $v = 5/24$.

Figure 7: The situation in the Corollary 3 and Corollary 4

the x'-coordinate $x'^{(i)}_\ell$ is situated within the interval $\mathrm{VR}^{(i)}_v$ ($r^{(i)}_\ell$ is the rectangle of maximum width in $W^{(i)}_L$).

Proof. Suppose by contradiction that $r^{(i)}_\ell$, does not intersect $\mathrm{VR}^{(i)}_v$, i.e. the x'-coordinate $x'^{(i)}_\ell$ is not within the interval $\mathrm{VR}^{(i)}_v$ (cf. Figure 7(b)). Let u be the successor of v in IN', i.e. $\mathrm{VL}^{(i)}_v = [v, u] \times [0, 1]$ and $\mathrm{VR}^{(i)}_v = [1 - u, 1 - v] \times [0, 1]$.

Since $r^{(i)}_\ell$ has the maximum width among the rectangles in $W^{(i)}_L$, no rectangle of $W^{(i)}_L$ intersects $\mathrm{VR}^{(i)}_v$. Thus all rectangles of $W^{(i)}_L$ have a bounded width of at most $1 - u$. The rectangles of $I^{(i)}_L$ that lie above the horizontal line at height y_1 or that intersect this line are bounded by rectangle r_1. Consequently, their total height is at least $1 - y_1 > 1/2$ and their widths are at most u. Thus, we satisfy Condition 1.7 with $y = 1/2$ and $x = u$, which is a contradiction

□

We also adopt this corollary to the bins turned by $90°, 180°$ and $270°$ and obtain the following structural theorem of the packing in the remaining bins in our solution (cf. Figure 8).

Theorem 3. *Let there be a packing in one bin B_i of our solution for which Lemma 2 and Lemma 7 are not applicable. It follows that values $h^{(i)}, v^{(i)} \in \mathrm{IN}$ with the following conditions exist:*

1.11. The sets $L^{(i)}_U, L^{(i)}_B, W^{(i)}_L$ and $W^{(i)}_R$ are non-empty and disjoint.

1.12. All rectangles in $L^{(i)}_U \setminus r^{(i)}_{u\ell}$ and $L^{(i)}_B \setminus r^{(i)}_{br}$ are completely situated within $\mathrm{VL}^{(i)}_{v^{(i)}}$ and $\mathrm{VR}^{(i)}_{v^{(i)}}$, respectively.

1.13. All rectangles in $W^{(i)}_L \setminus r^{(i)}_{b\ell}$ and $W^{(i)}_R \setminus r^{(i)}_{u\ell}$ are completely situated within $\mathrm{HB}^{(i)}_{h^{(i)}}$ and $\mathrm{HU}^{(i)}_{h^{(i)}}$, respectively.

1.14. If $r^{(i)}_{u\ell} \in L^{(i)}_U$, then the x'-coordinate $x'^{(i)}_{u\ell}$ is situated within $\mathrm{VL}^{(i)}_{v^{(i)}}$;
if $r^{(i)}_{br} \in L^{(i)}_B$, then the x-coordinate $x^{(i)}_{br}$ is situated within $\mathrm{VR}^{(i)}_{v^{(i)}}$.

1.15. If $r_{b\ell}^{(i)} \in W_L^{(i)}$, then the y'-coordinate $y_{b\ell}'^{(i)}$ is situated within $\mathrm{HB}_{h^{(i)}}^{(i)}$;
if $r_{ur}^{(i)} \in W_R^{(i)}$, then the y-coordinate $y_{ur}^{(i)}$ is situated within $\mathrm{HU}_{h^{(i)}}^{(i)}$.

1.16. the y-coordinates $y_u^{(i)}$ and $y_b'^{(i)}$ are situated within $\mathrm{HB}_{h^{(i)}}^{(i)}$ and $\mathrm{HU}_{h^{(i)}}^{(i)}$, respectively; the x-coordinates $x_r^{(i)}$ and $x_\ell'^{(i)}$ are situated within $\mathrm{VL}_{v^{(i)}}^{(i)}$ and $\mathrm{VR}_{v^{(i)}}^{(i)}$, respectively.

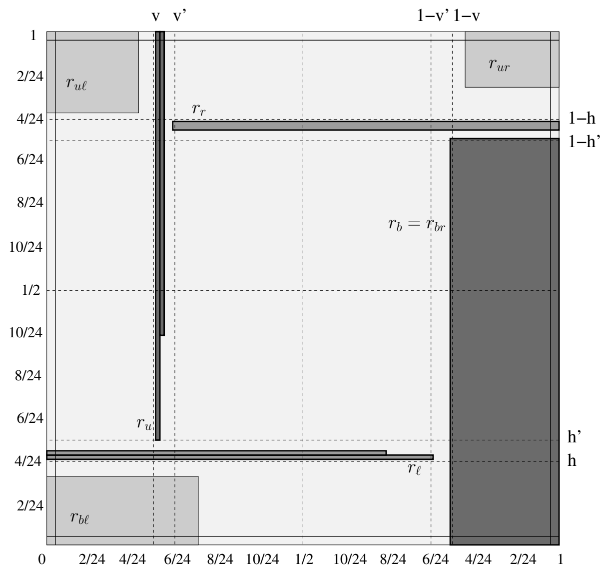


Figure 8: A packing as described in Theorem 3, with $v = 5/24$, $h = 4/24$ and $r_{br}^{(i)} \in L_B^{(i)}$.

In the following, we classify the remaining bins B_i, for which Lemma 2 and Lemma 7 are not applicable according to the values $v \in \mathrm{IN}$ and $h \in \mathrm{IN}$. Therefore, denote the values for which Theorem 3 for bin B_i holds by $h^{(i)} \in \mathrm{IN}$ and $v^{(i)} \in \mathrm{IN}$. Furthermore, let $h'^{(i)} \in \mathrm{IN}'$ and $v'^{(i)} \in \mathrm{IN}'$ be the successor of $h^{(i)}$ and $v^{(i)}$, respectively.

The packing that is described in Theorem 3 consists of almost five different regions. The region at the left of the rectangle $r_u^{(i)}$, the region below $r_\ell^{(i)}$, the region to the right of $r_b^{(i)}$, the region on top of $r_r^{(i)}$ and the region in the middle of the bin. There are only few rectangles that intersect two of these regions, since they have to lie completely inside the horizontal strips $\mathrm{HB}_{h^{(i)}}^{(i)}$ or $\mathrm{HU}_{h^{(i)}}^{(i)}$ or inside the vertical strips $\mathrm{VL}_{v^{(i)}}^{(i)}$ or $\mathrm{VR}_{v^{(i)}}^{(i)}$. We make use of this structure in the following section and remove the rectangles from two of the regions in order to remove a horizontal or vertical strip.

Classify the Bins with Rotations Before we continue with our analysis, we first state the differences in the version with rotations. In the proof of Lemma 7 we use 30 sets of bins $V_{x,1/2}, H_{x,1/2}, V_0$ and H_0, for $x \in \mathrm{IN}$. We rotate the packing of one bin B_i that is in a set $H_{x,1/2}$ or in H_0. The packing of this bin satisfies either Condition 1.7 or Condition 1.8. Therefore, it belongs to the set $V_{x,1/2}$ or V_0, respectively. Thus, when we are allowed to rotate, we have only 15 different sets of bins. We use Lemma 5 on each sequence of two bins in each set. If all 15 sets have an odd cardinality, we pack k bins, each satisfying at least one of the conditions Condition 1.7-1.10, into $3/2(k-15) + 2 \cdot 15 = 3/2k + 15/2 < 3/2k + 8$ bins.

Lemma' 7. *Let k denote the number of bins B_i in our solution for which an $x \in \mathrm{IN}$ and a $y \in \{0, 1/2\}$ exist so that one of the Conditions 1.7-1.10 holds. We are able to round up the* *rectangles of these k bins and rearrange them into $3/2k + 8$ bins, while the packing of each of the bins satisfies Property 1.*

2.2 Case Analysis

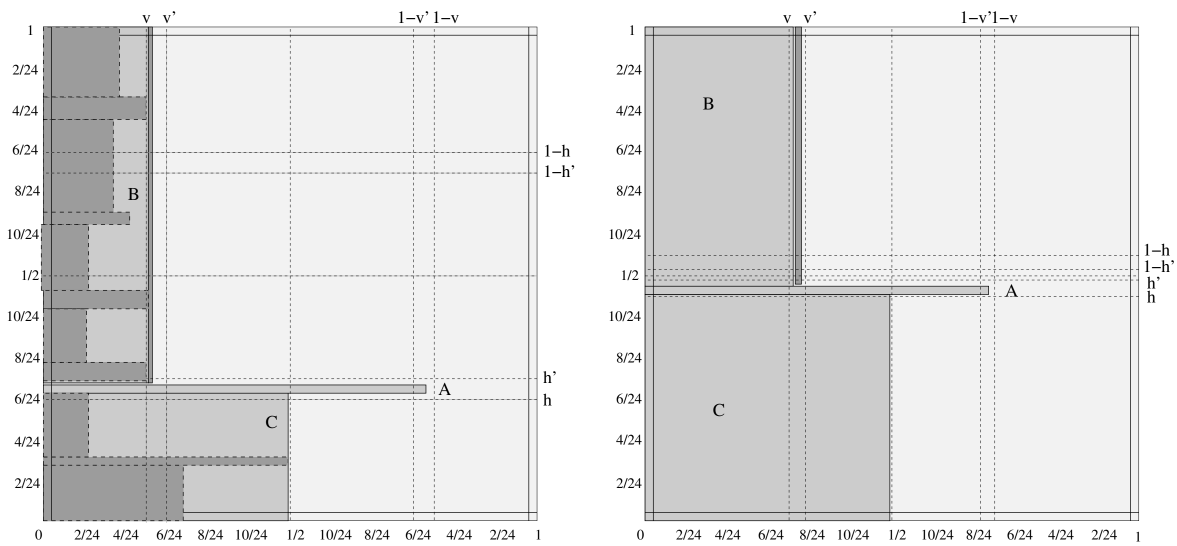
In the following, we suppose that for every bin there are values $v^{(i)} \in \mathrm{IN}$ and $h^{(i)} \in \mathrm{IN}$ so that Theorem 3 holds. We do a case analysis for the values $h^{(i)}$ and $v^{(i)}$.

Lemma 8. *Let $B_1, \ldots, B_k$ be k bins so that each bin B_i, for $i \in \{1, \ldots, k\}$, has a packing with the following conditions:*

2.1. $r_{b\ell}^{(i)} \notin W_L^{(i)}$ *or* $r_{ur}^{(i)} \notin W_R^{(i)}$,
2.2. $h^{(i)} \in \mathrm{IN}$,
2.3. $v^{(i)} \in \{0/24, \ldots, 7/24, 8/24 - \varepsilon_c\}$.

It follows that we are able to round up the rectangles in these bins and rearrange them into $3/2k + 2$ bins, while the packing of each of them satisfies either Property 1 or Property 2.

Proof. Let $i \in \{1, \ldots, k\}$. W.l.o.g. we assume that $r_{b\ell}^{(i)} \notin W_L^{(i)}$ since we are able to turn the bin by $180°$. We clear the strip $S_L^{(i)}$ in each of the k bins by using the property that the rectangles $W_L^{(i)}$ of width larger than $1/2$ are completely situated within $\mathrm{HB}_{h^{(i)}}^{(i)}$ and thus have a total height of at most $1/24$ (cf. Figure 9).



(a) Bin with $h = 7/24$ and $v = 5/24$ (b) Bin with $h = 11/24$ and $v = 7/24$

Figure 9: Two possible initial situations of Lemma 8.

We divide the rectangles in $I_L^{(i)}$ into three sets. Let $A^{(i)} = W_L^{(i)}$ be the set of rectangles that have a width of larger than $1/2$. Let $B^{(i)} \subset I_L^{(i)}$ denote the set of rectangles that have a width of at most $1/3$. Finally, let $C^{(i)}$ denote the set of the remaining rectangles in $I_L^{(i)}$ that have a width within $(1/3, 1/2]$. As mentioned above, the total height of the rectangles in $A^{(i)}$ is at most $1/24$, since $r_{b\ell}^{(i)} \notin W_L^{(i)}$ (Condition 2.1). We pack these rectangles on top of each other into a container of height $h(A^{(i)}) \leq 1/24$ and width $w_{\max}(A^{(i)}) \leq 1$. We treat this container as a rectangle $r_A^{(i)}$ of width $w_A^{(i)} = 1$ and height $h_A^{(i)} = h(A^{(i)})$.

The rectangles in $L_U^{(i)}$ have their x'-coordinates within $\mathrm{VL}_{v^{(i)}}^{(i)}$. Thus, the rectangles of $I_L^{(i)}$ that lie on the left of the rectangles in $L_U^{(i)}$, including $r_{u\ell}^{(i)}$ if $r_{u\ell}^{(i)} \in L_U^{(i)}$, have a bounded width of $v'^{(i)} \leq 8/24 = 1/3$ (Condition 2.3). Remember that $v'^{(i)}$ is the successor of $v^{(i)}$ in IN'. Therefore, these rectangles belong to the set $B^{(i)}$. Consequently, the total height of the rectangles in the set $B^{(i)}$ is at least $h(B^{(i)}) = h_u^{(i)} > 1/2$. The rectangles are also packed on top of each other into the container/rectangle $r_B^{(i)}$ of a width $w_B^{(i)} = w_{\max}(B^{(i)}) \leq 1/3$ and a height $h_B^{(i)} = h(B^{(i)}) > 1/2$. Moreover, the rectangles in $C^{(i)}$ are packed on top of each other into a container/rectangle $r_C^{(i)}$ of a width $w_C^{(i)} = w_{\max}(C^{(i)}) \leq 1/2$ and a height $h_C^{(i)} = h(C^{(i)}) \leq 1 - h(B^{(i)}) = 1 - h_B^{(i)}$.

We clear the strips $S_L^{(i)}$ of each bin by packing the rectangles of each sequence of 6 bins into 3 additional bins C_1, C_2, C_3. Let $B_1, \dots, B_6$ be 6 bins among the k bins. W.l.o.g. we assume that these bins are sorted by non-decreasing heights of $h_B^{(i)}$.

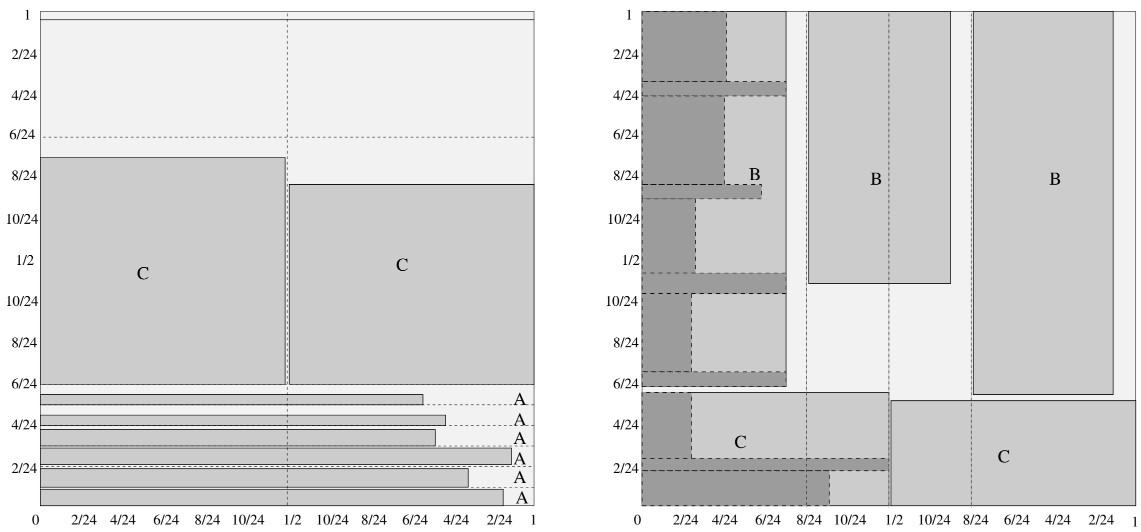
The rectangles $r_A^{(1)}, \dots, r_A^{(6)}$ have a total height of at most $6 \cdot 1/24 = 1/4$. We pack them on top of each other at the bottom of bin C_1 with their x-coordinates positioned on the value 0. On top of these rectangles we pack $r_C^{(1)}$ and $r_C^{(2)}$. They have both a width and a height of at most $1/2$ and fit next to each other on the positions $(0, 1/4)$ and $(1/2, 1/4)$. The uppermost horizontal strip of the height $1/4 \geq \varepsilon_c$ is still free of rectangles (cf. Figure 10(a)). We employ Lemma 4 on this bin in order to round up the heights.

The rectangles $r_B^{(i)}$ and $r_C^{(i)}$ always fit on top of each other since $h_C^{(i)} < 1 - h_B^{(i)}$. This allows us to place $r_C^{(3)}$ and $r_B^{(3)}$ on top of each other in bin C_2 with their x-coordinates positioned on the value 0. The rectangles $r_C^{(4)}$ and $r_B^{(4)}$ are also placed on top of each other, where $r_C^{(4)}$ is placed on position $(1/2, 0)$ and $r_B^{(4)}$ is placed on top of $r_C^{(4)}$ on the x-coordinate $2/3$. Between the rectangles $r_B^{(3)}$ and $r_B^{(4)}$ there is a free space of width $1/3$ and height at least $\min\{h_B^{(3)}, h_B^{(4)}\} = h_B^{(3)}$. Since $h_B^{(1)} \leq h_B^{(3)}$ and $w_B^{(1)} \leq 1/3$ this space is sufficient to place $r_B^{(1)}$ on top of $r_C^{(3)}$ and $r_C^{(4)}$ on the x-coordinate $1/3$ (cf. Figure 10(b)).

A horizontal or vertical strip free of rectangles does not necessarily have to exists in this bin. However, we are able to round up the widths of the rectangles $r_B^{(1)}, r_B^{(3)}$ and $r_B^{(4)}$ to the next largest multiple of $\varepsilon_c^2/2$ that is at most $1/3$. The widths of the rectangles $r_C^{(3)}$ and $r_C^{(4)}$ are also rounded to the next largest multiple of $\varepsilon_c^2/2$ that is at most $1/2$. The rectangles that are inside the rectangles $r_B^{(i)}$ and $r_C^{(i)}$ are packed on top of each other. This enables us also to round up their widths to the next largest multiple of $\varepsilon_c^2/2$, which is at most $1/3$ or $1/2$, respectively. Furthermore, their x-coordinates are either $0, 1/3, 1/2, 2/3$ and hence multiples of $\varepsilon_c^2/2$. Consequently, this packing satisfies Property 1. The packing of bin C_3 is analogous to the packing of C_2 with rectangles $r_C^{(5)}, r_C^{(6)}, r_B^{(2)}, r_B^{(5)}$ and $r_B^{(6)}$.

We modify each sequence of 6 of the k bins. If $\ell \leq 4$ bins remain, we pack the rectangles that intersect S_L into ℓ additional bins. In this case, we need in total $3/2(k-\ell) + 2\ell = 3/2k + \ell/2 \leq 3/2k+2$. If $\ell = 5$ bins remain, we adopt the same packing as described above without the rectangles $r_A^{(6)}, r_B^{(6)}$ and $r_C^{(6)}$. We need in total $3/2(k-\ell)+\ell+3 = 3/2k-(3/2 \cdot 5)+8 \leq 3/2k+1$. The case analysis in this paragraph is also used in some of the following lemmas, we do not repeat it there. □

Analogously, we achieve the same result for the following corollary. By turning the bin by $90°$, the proof is exactly the same. Consequently, we do not need this lemma, if we are allowed to rotate the rectangles.



(a) Structure of the packing in bin C_1 (b) Structure of the packing in bin C_2 and C_3

Figure 10: The structure of the additional bins of Lemma 8

Lemma 9. *Let $B_1, \ldots, B_k$ be k bins so that each bin B_i, for $i \in \{1, \ldots, k\}$, has a packing with the following conditions:*

2.4. $r_{u\ell}^{(i)} \notin L_U^{(i)}$ *or* $r_{br}^{(i)} \notin L_B^{(i)}$,

2.5. $h^{(i)} \in \{0/24, \ldots, 7/24, 8/24 - \varepsilon_c\}$,

2.6. $v^{(i)} \in \mathrm{IN}$.

It follows that we are able to round up the rectangles in these bins and rearrange them into $3/2k + 2$ bins, while the packing of each of them satisfies either Property 1 or Property 2.

The following lemma covers the case that there is a bin B_i with $r_{u\ell}^{(i)} \in L_U^{(i)}, r_{br}^{(i)} \in L_B^{(i)}, r_{ur}^{(i)} \in W_R^{(i)}$ and $r_{b\ell}^{(i)} \in W_L^{(i)}$.

Lemma 10. *Let $B_1, \ldots, B_k$ be k bins so that each bin B_i, for $i \in \{1, \ldots, k\}$, has a packing with the following conditions:*

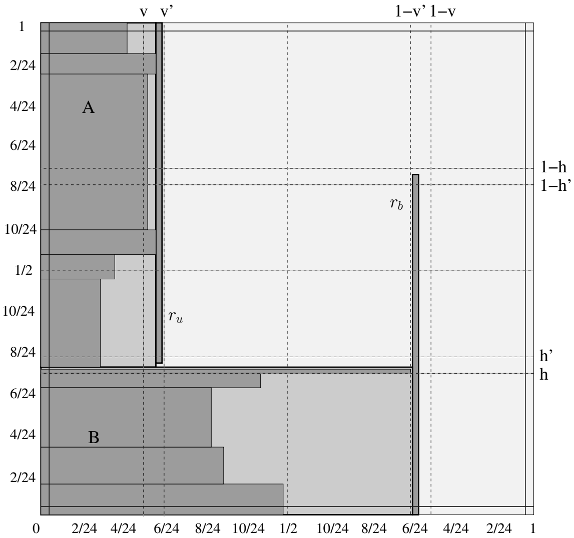
2.7. $h^{(i)} \in \{0/24, \ldots, 7/24\}$,

2.8. $v^{(i)} \in \{0/24, \ldots, 7/24, 8/24 - \varepsilon_c\}$.

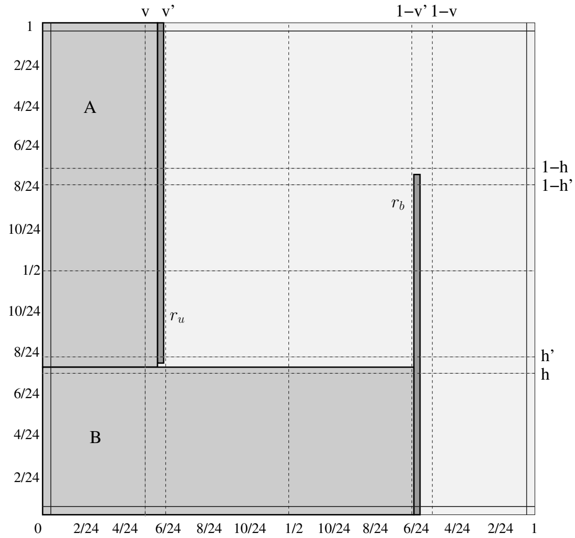
It follows that we are able to round up the rectangles in these bins and rearrange them into $3/2k + 1$ bins, while the packing of each of them satisfies either Property 1 or Property 2.

Proof. Let $i \in \{1, \ldots, k\}$. W.l.o.g. we assume that $x_u'^{(i)} \leq 1 - x_b^{(i)}$ since we are able to turn the bin by $180°$. We want to remove the rectangles intersecting the strip $S_L^{(i)}$ in each bin.

Similar to the proof of Lemma 8 we use containers/rectangles for grouping the rectangles in $I_L^{(i)}$. If $r_{u\ell}^{(i)} \in L_U^{(i)}$ and $r_{u\ell}^{(i)} = r_u^{(i)}$, i.e. the rectangle in the upper left corner is the largest rectangle in $L_U^{(i)}$, the rectangle $r_A^{(i)}$ is $r_{u\ell}^{(i)}$. In any other case, let $A^{(i)} \subset I_L^{(i)}$ be the set of rectangles that are at the left of $r_u^{(i)}$, i.e. the rectangles in $I_L^{(i)}$ that lie above the horizontal line at height $y_u^{(i)}$ and that intersect with it. In this case, we define $r_A^{(i)}$ as a rectangle of height



(a) Rectangles intersecting $S_L^{(i)}$



(b) Construction of rectangles $r_A^{(i)}$ and $r_B^{(i)}$

Figure 11: A possible initial situation of Lemma 10

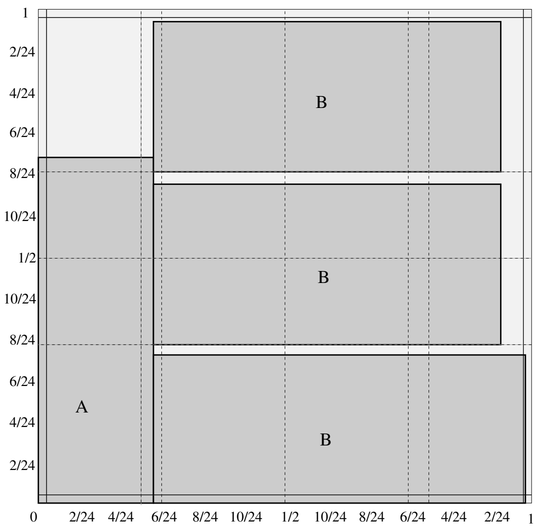
$h_A^{(i)} = h(A^{(i)})$ and width $w_A^{(i)} = w_{\max}(A^{(i)})$. The width of $r_A^{(i)}$ is limited in both cases to at most $w_A^{(i)} \leq x_u'^{(i)} \leq v'^{(i)} \leq 8/24 = 1/3$ (Condition 2.8). Since $r_u^{(i)}$ intersects the horizontal interval $\mathrm{HB}_{h^{(i)}}^{(i)}$, the height of $r_A^{(i)}$ is at least $h_A^{(i)} \geq 1 - h'^{(i)} \geq 1 - (8/24 - \varepsilon_c) = 16/24 + \varepsilon_c = 2/3 + \varepsilon_c$ (Condition 2.7).

The widths of the rectangles in $W_L^{(i)}$ are bounded by the leftmost rectangle of $L_B^{(i)}$. Hence, their widths are at most $x_b^{(i)} \leq 1 - x_u'^{(i)}$. We use a set $B^{(i)} = I_L^{(i)} \setminus A^{(i)}$ of the remaining rectangles that intersect $S_L^{(i)}$. Analogously, we define a rectangle $r_B^{(i)}$ for these rectangles of width $w_B^{(i)} = w_{\max}(B^{(i)}) \leq x_b^{(i)} \leq 1 - x_u'^{(i)}$ and height $h_B^{(i)} = 1 - h_A^{(i)} \leq 1/3 - \varepsilon_c$. Note that $w_A^{(i)} + w_B^{(i)} \leq x_u'^{(i)} + (1 - x_u'^{(i)}) = 1$ and therefore the rectangles $r_A^{(i)}$ and $r_B^{(i)}$ fit next to each other in one bin (cf. Figure 11).

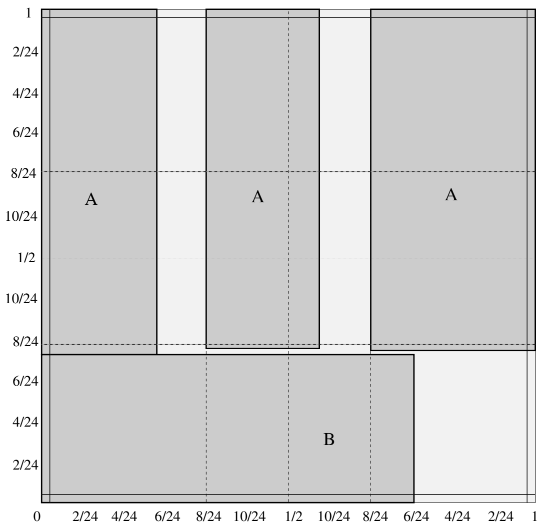
In order to employ Lemma 3, we pack $r_A^{(i)}$ and $r_B^{(i)}$ and with them all rectangles that intersect with $S_L^{(i)}$ into an additional bin. To this end, we pack the rectangles of each sequence of 4 bins $B_1, \ldots, B_4$ of the k bins into 2 additional bins C_1 and C_2. W.l.o.g. we assume that $r_A^{(1)}$ is the rectangle with the minimum width among the rectangles $r_A^{(1)}, \ldots, r_A^{(4)}$ and $r_B^{(2)}$ is the rectangle with the minimum height among the rectangles $r_B^{(2)}, r_B^{(3)}, r_B^{(4)}$.

We pack $r_A^{(1)}$ in the lower left corner of C_1 on the position $(0, 0)$. Since $w_A^{(1)} \leq w_A^{(3)} \leq 1 - w_B^{(3)}$ and $w_A^{(1)} \leq w_A^{(4)} \leq 1 - w_B^{(4)}$, we are able to pack the rectangles $r_B^{(1)}, r_B^{(3)}$ and $r_B^{(4)}$ on the right side of $r_A^{(1)}$. These three rectangles each have a height of at most $1/3 - \varepsilon_c$ and thus we are able to place them on top of each other on the positions $(w_A^{(1)}, 0), (w_A^{(1)}, 1/3)$ and $(w_A^{(1)}, 2/3)$. On top of $r_B^{(4)}$ there is still a free space of height ε_c. The rectangle $r_A^{(1)}$ also has a height of at most $1 - \varepsilon_c$, as there would otherwise be no rectangle in $W_L^{(1)}$. Consequently, the uppermost strip of height ε_c is free and we are able to employ Lemma 3 on C_1 (cf. Figure 12(a))

The remaining rectangles $r_A^{(2)}, r_A^{(3)}, r_A^{(4)}$ and $r_B^{(2)}$ have to be packed into bin C_2. It holds that $h_A^{(i)} + h_B^{(i)} = 1$ and therefore $h_B^{(2)} \leq h_B^{(3)} = 1 - h_A^{(3)}$ and $h_B^{(2)} \leq h_B^{(4)} = 1 - h_A^{(4)}$. Thus, the rectangles $r_A^{(2)}, r_A^{(3)}$ and $r_A^{(4)}$ each fit above $r_B^{(2)}$. The widths of $r_A^{(2)}, r_A^{(3)}$ and $r_A^{(4)}$ are at most



(a) Structure of the packing in bin C_1



(b) Structure of the packing in bin C_2

Figure 12: The structure of the additional bins of Lemma 10

$1/3$. Hence, we are able to place $r_B^{(2)}$ on the position $(0,0)$, $r_A^{(2)}$ on the position $(0, h_B^{(2)})$, $r_A^{(3)}$ on the position $(1/3, h_B^{(2)})$ and $r_A^{(4)}$ on the position $(2/3, h_B^{(2)})$(cf. Figure 12(b)). A horizontal or vertical strip free of rectangles does not necessarily have to exist in this bin. However, there is no rectangle on the right of $r_B^{(2)}$ and therefore we are able to round up the rectangle $r_B^{(2)}$ and the rectangles inside it to the next largest multiple of $\varepsilon_c^2/2$. The rectangles $r_A^{(2)}, r_A^{(3)}$ and $r_A^{(4)}$ are positioned on x-coordinates $0, 1/3$ and $2/3$ that are multiples of $\varepsilon_c^2/2$. We are able to round up the widths of these rectangles to the next largest multiple of $\varepsilon_c^2/2$ to at most $1/3$. If $r_A^{(i)}$ is a container, the rectangles within it are therefore also positioned on a multiple of $\varepsilon_c^2/2$ and can be rounded up to the next largest multiple of $\varepsilon_c^2/2$. Consequently, this packing satisfies Property 1.

We do this for each sequence of 4 of the k bins. Let $\ell \leq 3$ denote the number of remaining bins. If $\ell \leq 2$ we employ an additional bin for each of the ℓ bins and we pack the rectangles of the strip $S_L^{(i)}$ into it. We have $3/2(k-\ell)+2\ell = 3/2\cdot k+\ell/2 \leq 3/2\cdot k+1$ bins in total. If $\ell = 3$ we pack them according to the method described above without the rectangles $r_A^{(4)}$ and $r_B^{(4)}$ and use 2 additional bins. We obtain in this case $3/2(k-\ell)+\ell+2 = 3/2\cdot k-\ell/2+2 \leq 3/2\cdot k+1$ bins. $\square$

The lemma described above does not work for $h^{(i)} = 8/24-\varepsilon_c$ since the rectangles $r_B^{(i)}$ might have a height close to $1/3$ and hence the uppermost strip in bin C_1 is not free. Therefore, we have to use a slight modification.

Lemma 11. *Let $B_1, \ldots, B_k$ be k bins so that each bin B_i, for $i \in \{1, \ldots, k\}$, has a packing with the following conditions:*

2.9. $r_{ur}^{(i)} \in W_R^{(i)}$ *and* $r_{b\ell}^{(i)} \in W_L^{(i)}$,

2.10. $h^{(i)} = 8/24 - \varepsilon_c$,

2.11. $v^{(i)} \in \{0/24, \ldots, 7/24, 8/24-\varepsilon_c\}$.

It follows that we are able to round up the rectangles in these bins and rearrange them into $(3/2+\varepsilon_c)\cdot k+2$ bins, while the packing of each of them satisfies either Property 1 or Property 2.

Proof. Let $i \in \{1,\ldots,k\}$. We use the same packing as in the proof of Lemma 10. The

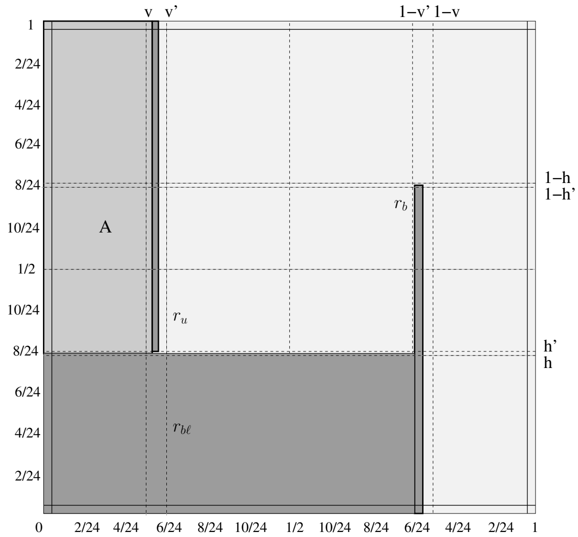


Figure 13: Initial packing of Lemma 11

difference is that, after possibly turning the bin by $180°$, we have $r_{b\ell}^{(i)} \in W_L^{(i)}$ and hence the y'-coordinate of $r_{b\ell}^{(i)}$ is within $\mathrm{HB}_{8/24-\varepsilon_c}^{(i)}$ (Condition 2.10). Furthermore, the rectangle $r_u^{(i)}$ also intersects $\mathrm{HB}_{8/24-\varepsilon_c}^{(i)}$ and has its y-coordinate within it. All rectangles in $I_L^{(i)}$ that are above the horizontal line at height $y_u^{(i)}$ and that intersect it, belong to $A^{(i)}$. Consequently, the rectangles in $B^{(i)}$ consist of $r_{b\ell}^{(i)}$ and rectangles of a total height of at most ε_c. We move these rectangles of the total height at most ε_c into additional bins by packing them on top of each other at the x-coordinate 0. For all k bins, we need at most $\lceil \varepsilon_c \cdot k \rceil \le \varepsilon_c \cdot k + 1$ additional bins. We are able to round up the widths to the next largest multiple of $\varepsilon_c^2/2$ and satisfy Property 1.

At this moment, we have $B^{(i)} = \{r_{b\ell}^{(i)}\}$ and thus $r_B^{(i)} = r_{b\ell}^{(i)}$. We adopt the same packing as in the proof of Lemma 10. We round the heights of the rectangles $r_{b\ell}^{(1)}, r_{b\ell}^{(3)}$ and $r_{b\ell}^{(4)}$ in the bin C_1 to the next largest multiple of $\varepsilon_c^2/2$ which is at most $1/3$. The rounding of the remaining rectangles is the same as in the proof of Lemma 10. In total, we have $(3/2+\varepsilon_c)\cdot k+2$ bins. □

This finishes the case analysis for the values $v^{(i)} < 8/24 = 1/3$ and $h^{(i)} < 8/24 = 1/3$. What is left is the case in that the rectangles of a height larger than $1/2$ or the rectangles of a width larger than $1/2$ are situated close to the middle of the bin.

2.2.1 Intervals in the Middle

If we have $h^{(i)} \in \{0/24,\ldots,7/24, 8/24-\varepsilon_c\}$ and $v^{(i)} \in \{0/24,\ldots,7/24, 8/24-\varepsilon_c\}$, we are able to use one of the lemmas above. Hence, the packing in the remaining bins has at least one of the two values $v^{(i)}$ and $h^{(i)}$ in the set $\{8/24,\ldots,11/24, 12/24-\varepsilon_c\}$.

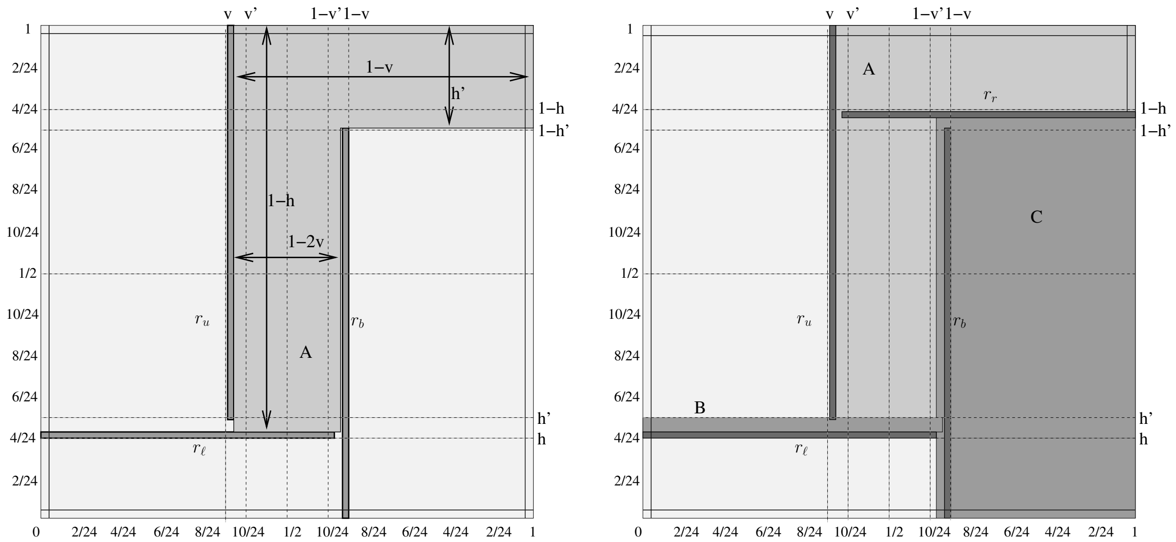
Lemma 12. *Let $B_1,\ldots,B_k$ be k bins so that each bin B_i, for $i \in \{1,\ldots,k\}$, has a packing with the following conditions:*

2.12. $v^{(i)} \in \{8/24, 9/24, 10/24, 11/24\}$,

2.13. $h^{(i)} \in \{2/24,\ldots,9/24\}$.

It follows that we are able to round up the rectangles in these bins and rearrange them into $3/2k + 1$ bins, while the packing of each of them satisfies either Property 1 or Property 2.

Proof. Let $i \in \{1, \dots, k\}$. We want to move the rectangles in the middle of the bin B_i to an additional bin, in order to free $S_R^{(i)}$. Let w.l.o.g. $y_\ell'^{(i)} \leq 1 - y_r^{(i)}$ since we are able to turn the bin by 180°.



(a) Definition of region $A^{(i)}$

(b) Definition of region $B^{(i)}$ and $C^{(i)}$

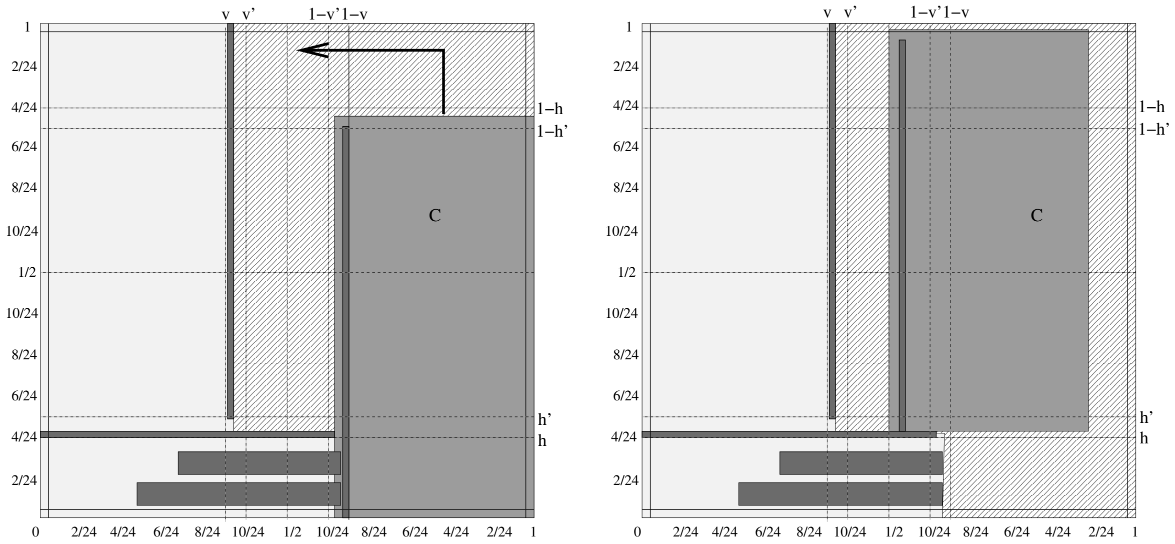
Figure 14: The definition of the regions of Lemma 12

Define a non-rectangular, Γ-shaped region $A^{(i)}$, consisting of the rectangles in the middle and upper right side of B_i. We define this region along the rectangles $r_u^{(i)}, r_\ell^{(i)}, r_b^{(i)}$ and the right and top of the bin, in order to ensure that there are only few rectangles that intersect this region from the sides. The region $A^{(i)}$ is defined by the coordinates $(x_u'^{(i)}, y_\ell'^{(i)}), (x_b^{(i)}, y_\ell'^{(i)}), (x_b^{(i)}, y_b'^{(i)}),$ $(1, y_b'^{(i)}), (1, 1)$ and $(x_u'^{(i)}, 1)$ (cf. Figure 14(a)). We treat this region with all rectangles that are completely situated inside it as one object $o_A^{(i)}$ and move this object to an additional bin. The heights and widths are bounded as follows:

The longer, left side has a height of $1 - y_\ell'^{(i)} \leq 1 - h^{(i)} \leq 1 - 2/24 = 22/24$, the shorter, right side a height of at most $1 - y_b'^{(i)} \leq 1 - (1 - h'^{(i)}) = h'^{(i)} \leq 10/24$ (Condition 2.12). The lower part of this object has a width of at most $x_b^{(i)} - x_u'^{(i)} \leq (1 - v^{(i)}) - v^{(i)} = 1 - 2v^{(i)} \leq 8/24 = 1/3$, the upper part one of $1 - x_u'^{(i)} \leq 1 - v^{(i)} \leq 1 - 8/24 = 16/24 = 2/3$ (Condition 2.13).

It is possible that there are rectangles that are not completely located inside this region, but intersect it from below or from the left. The rectangles that intersect it from the left have to be situated between the rectangles $r_\ell^{(i)}$ and $r_u^{(i)}$ since we defined the region $A^{(i)}$ along the right side of rectangle $r_u^{(i)}$. The rectangles $r_\ell^{(i)}$ and $r_u^{(i)}$ intersect both with $\mathrm{HB}_{h^{(i)}}^{(i)}$, hence the total height of these rectangles is bounded by $1/24$. We call the set of these rectangles $B^{(i)}$ and pack them into a container/rectangle $r_B^{(i)}$ of height $h_B^{(i)} = h(B^{(i)}) \leq 1/24$ and width $w_B^{(i)} = w_{\max}(B^{(i)}) \leq x_b^{(i)} \leq 1 - v^{(i)} \leq 1 - 8/24 = 16/24 = 2/3$ (cf. Figure 14(b)).

The rectangles that intersect $A^{(i)}$ from below are situated on the right of the x'-coordinate $x_\ell'^{(i)}$ since $A^{(i)}$ is defined along $r_\ell^{(i)}$. Furthermore, these rectangles are all bounded by the



(a) Moving $r_C^{(i)}$ on top of $r_\ell^{(i)}$

(b) The strip $S_R^{(i)}$ is completely free

Figure 15: Moving the region $C^{(i)}$ in the proof of Lemma 12

rectangle $r_r^{(i)}$ that is completely situated inside $A^{(i)}$. Therefore, we define a region $C^{(i)}$ that contains all rectangles that intersect $A^{(i)}$ from below by the coordinates $(x_\ell'^{(i)}, 0), (1, 0), (1, y_r^{(i)})$ and $(x_\ell'^{(i)}, y_r^{(i)})$. There is no rectangle that intersects $C^{(i)}$ from above since we defined it along the lower edge of $r_r^{(i)}$. The rectangles that intersect region $C^{(i)}$ from the left above the rectangle $r_\ell^{(i)}$ are completely situated inside $A^{(i)}$. The rectangles that intersect $C^{(i)}$ from the left below the rectangle $r_\ell^{(i)}$ are bounded by the rectangle $r_b^{(i)}$. Hence, they do not intersect $S_R^{(i)}$ and we do not move these rectangles. We treat the region $C^{(i)}$ as one container/rectangle $r_C^{(i)}$ of height $h_C^{(i)} = y_r^{(i)} \leq 1 - y_\ell'^{(i)}$ and width $w_C^{(i)} = 1 - x_\ell'^{(i)} \leq 1 - (1 - v'^{(i)}) \leq 12/24 - \varepsilon_c$ that contains all rectangles completely situated inside this region.

We move the objects $o_A^{(i)}$ and $r_B^{(i)}$ into an additional bin, while $r_C^{(i)}$ is moved inside B_i. Without these three objects the region $(x_u'^{(i)}, y_\ell'^{(i)}), (1, y_\ell'^{(i)}), (1, 1)$ and $(x_u'^{(i)}, 1)$ at the right side of $r_u^{(i)}$ is completely free of rectangles. It is $h_C^{(i)} \leq 1 - y_\ell'^{(i)}$ and $w_C^{(i)} \leq 12/24 - \varepsilon_c$. Thus, we can place $r_C^{(i)}$ on the position $(1/2, y_\ell'^{(i)})$ leaving the left strip $S_R^{(i)}$ completely free of rectangles (cf. Figure 15). We employ Lemma 3 on this bin in order to round up the rectangles and to satisfy Property 1.

Let there be two bins B_1 and B_2 of the k bins and let C_1 be an additional empty bin. We pack the objects $o_A^{(1)}, o_A^{(2)}, r_B^{(1)}$ and $r_B^{(2)}$ into C_1.

We place $r_B^{(1)}$ at the bottom of the bin C_1 on the position $(1/3, 0)$. The object $o_A^{(1)}$ is turned by $180°$ so that the long edge is at the bottom. We pack this object on top of $r_B^{(1)}$ at position $(1/3, 1/24)$. Both objects occupy the region $(1/3, 0), (1, 0), (1, 23/24), (2/3, 23/24), (2/3, 11/24)$ and $(1/3, 11/24)$.

The object $o_A^{(2)}$ is placed on top of $o_A^{(1)}$ with the top edge at height $22/24$. It occupies the region $(0, 0), (1/3, 0), (1/3, 12/24), (2/3, 12/24), (2/3, 22/24)$ and $(0, 22/24)$ (cf. Figure 16). These regions do not overlap. On top of $o_A^{(2)}$, there is still a free space of width $2/3$ and height $2/24$. In this space we place $r_B^{(2)}$ on position $(0, 22/24)$. It follows that there is a strip of the

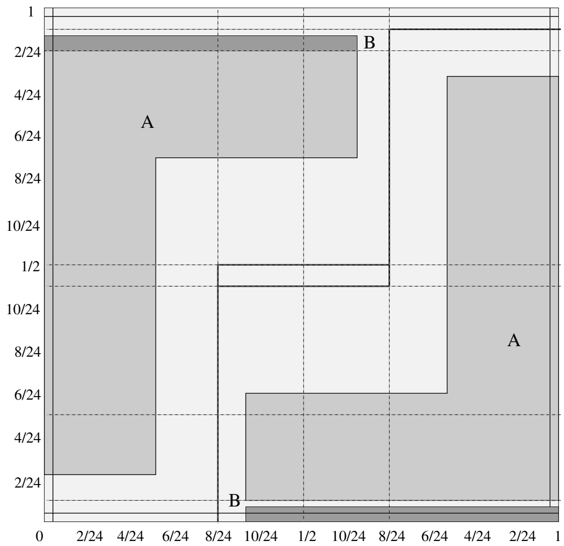


Figure 16: Packing of the additional bins in the proof of Lemma 12

height $1/24$ free of rectangles including the strip S_U. This allows us to use Lemma 4 in order to round up the rectangles and to satisfy Property 2.

We repeat this step with each sequence of 2 of the k bins and achieve a packing of $3/2k+1$ bins in total, when k is odd. □

The analogous lemma by exchanging the values $h^{(i)}$ and $v^{(i)}$ is as follows. However, in the version that allows rotation we do not need this lemma, turn the packing by $90°$ to adopt Lemma 12 instead.

Lemma 13. *Let $B_1, \dots, B_k$ be k bins so that each bin B_i, for $i \in \{1, \dots, k\}$, has a packing with the following conditions:*

2.14. $h^{(i)} \in \{8/24, 9/24, 10/24, 11/24\}$,
2.15. $v^{(i)} \in \{2/24, \dots, 9/24\}$.

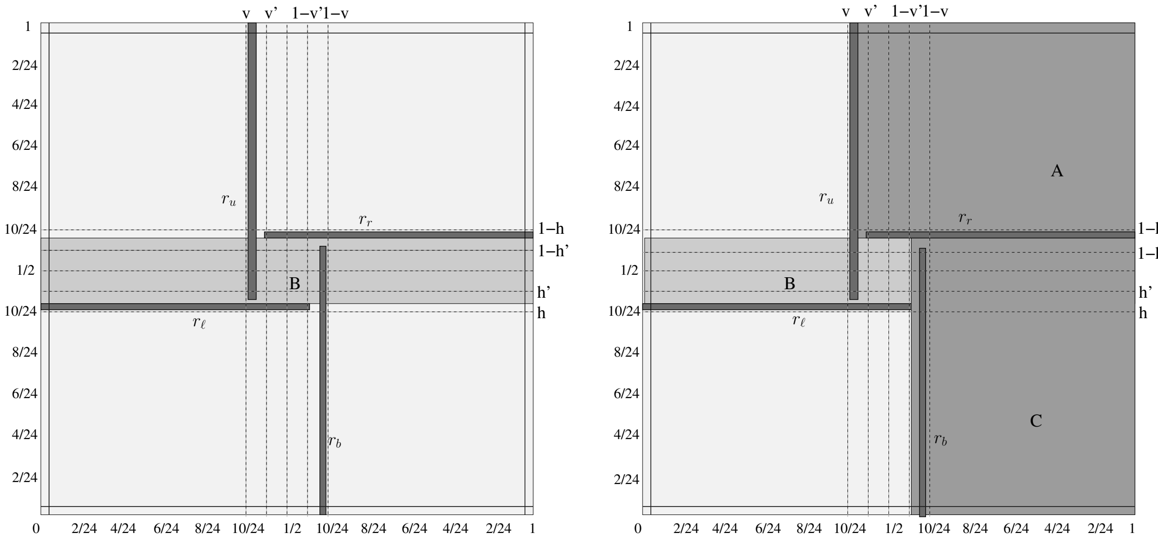
It follows that we are able to round up the rectangles in these bins and rearrange them into $3/2k+1$ bins, while the packing of each of them satisfies either Property 1 or Property 2.

In the following lemma we use the same technique as in the lemma above. We also move the region in the middle and in the upper right corner into an additional bin. The region in the lower right corner is shifted in the same way as in the Lemma 12 above. Furthermore, we use the fact that the values $v^{(i)}$ and $h^{(i)}$ are in the same range.

Lemma 14. *Let $B_1, \dots, B_k$ be k bins so that each bin B_i, for $i \in \{1, \dots, k\}$, has a packing with the following conditions:*

2.16. $h^{(i)} \in \{10/24, 11/24\}$,
2.17. $v^{(i)} \in \{10/24, 11/24\}$.

It follows that we are able to round up the rectangles in these bins and rearrange them into $3/2k+1$ bins, while the packing of each of them satisfies either Property 1 or Property 2.



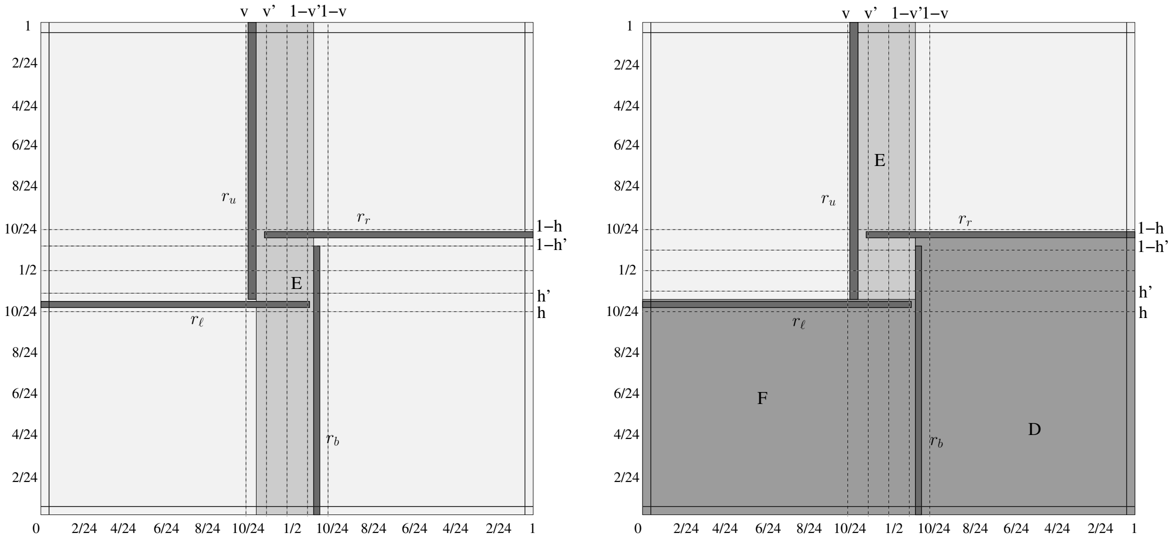
(a) Definition of the region $B^{(i)}$ (b) Definition of region $A^{(i)}$ and $C^{(i)}$

Figure 17: Definitions of the regions of the first $\lfloor k/2 \rfloor$ bins in the proof of Lemma 14

Proof. Let $i \in \{1, \dots, \lfloor k/2 \rfloor\}$ and $j \in \{\lfloor k/2 \rfloor + 1, \dots, k\}$ and let w.l.o.g. $y_\ell'^{(i)} \leq 1 - y_r^{(i)}$ and $x_b^{(j)} \leq 1 - x_u'^{(j)}$ since we are able to turn the bin B_i or B_j by 180°. We use a similar region definition as in Lemma 12.

Let $A^{(i)}$ be the region defined by $(x_u'^{(i)}, y_r^{(i)}), (1, y_r^{(i)}), (1, 1)$ and $(x_u'^{(i)}, 1)$. The region $B^{(i)}$ in the middle of bin B_i is defined by $(0, y_\ell'^{(i)}), (1, y_\ell'^{(i)}), (1, y_r^{(i)})$ and $(0, y_r^{(i)})$. The region $C^{(i)}$ is defined by $(x_\ell'^{(i)}, 0), (1, 0), (1, y_r^{(i)})$ and $(x_\ell'^{(i)}, y_r^{(i)})$ (cf. Figure 17). Again, we treat these regions as containers/rectangles. $r_A^{(i)}$ has height $h_A^{(i)} \leq 1 - y_r^{(i)} \leq 1 - (1 - h'^{(i)}) = h'^{(i)} \leq 12/24 - \varepsilon_c$ (Condition 2.17) and width $w_A^{(i)} \leq 1 - x_u'^{(i)} \leq 1 - v^{(i)} \leq 1 - 10/24 = 14/24$ (Condition 2.16). The width of $r_B^{(i)}$ is $w_B^{(i)} = 1$ and the height is $h_B^{(i)} = y_r^{(i)} - y_\ell'^{(i)} \leq (1 - h^{(i)}) - h^{(i)} = 1 - 2h^{(i)} \leq 1 - 20/24 = 4/24$ (Condition 2.16). The rectangle $r_C^{(i)}$ has width $w_C^{(i)} = 1 - x_\ell'^{(i)} \leq 1 - (1 - h'^{(i)}) = h'^{(i)} \leq 12/24 - \varepsilon_c$ (Condition 2.16) and height $h_C^{(i)} = y_r^{(i)} \leq 1 - y_\ell'^{(i)}$. All rectangles that intersect the region $B^{(i)}$ from below are situated at the right of the rectangle $r_\ell^{(i)}$. Hence, these rectangles are all completely situated inside the region $C^{(i)}$. The rectangles that intersect $B^{(i)}$ from above have to be on the left of rectangle $r_r^{(i)}$. Thus, they are completely situated at the left of the vertical line at x-coordinate $1/2$. These rectangles are not moved. This holds especially for the rectangles that intersect the region $A^{(i)}$ from below between the rectangles $r_u^{(i)}$ and $r_r^{(i)}$. There are no further rectangles that intersect $A^{(i)}$ while they are not completely situated inside this region. The rectangles that intersect region $C^{(i)}$ are either completely situated inside region $B^{(i)}$ or are situated below $r_\ell^{(i)}$ and thus bounded by rectangle $r_b^{(i)}$. The values of the x'-coordinates of these rectangles is at most $x_b^{(i)}$. We do not move these rectangles.

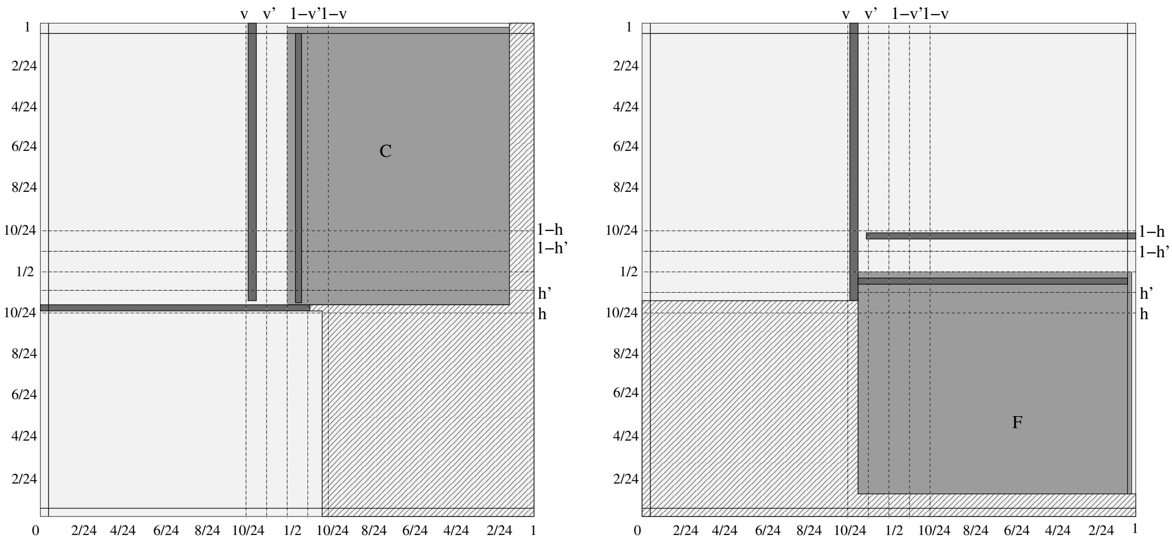
We define similar regions in the bin B_j. The only difference is that they are rotated by 90° (cf. Figure 18). Let $D^{(j)}$ be the region defined by the coordinates $(x_b^{(j)}, 0), (1, 0), (1, y_r^{(j)})$ and $(x_b^{(j)}, y_r^{(j)})$, $E^{(j)}$ is defined by the coordinates $(x_u'^{(j)}, 0), (x_b^{(j)}, 0), (x_b^{(j)}, 1)$ and $(x_u'^{(j)}, 1)$. The region $F^{(j)}$ is defined by the coordinates $(0, 0), (x_b^{(j)}, 0), (x_b^{(j)}, y_u^{(j)})$ and $(0, y_u^{(j)})$. The values for the heights and the widths of the corresponding rectangles $r_D^{(j)}, r_E^{(j)}$ and $r_F^{(j)}$ are the same as



(a) Definition of region $E^{(i)}$

(b) Definition of region $D^{(i)}$ and $F^{(i)}$

Figure 18: Definitions of the regions of the last $\lceil k/2 \rceil$ bins in the proof of Lemma 14

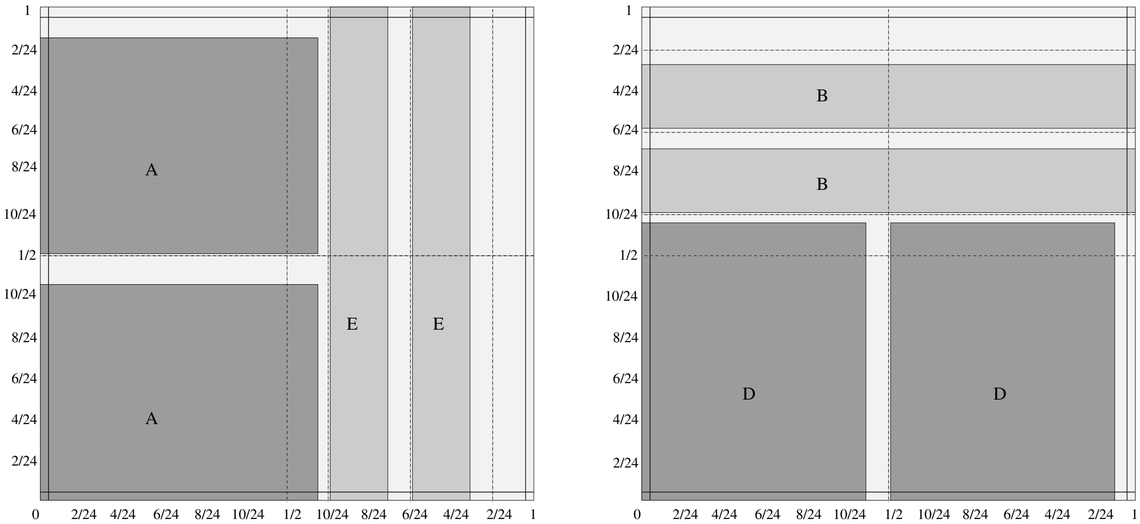


(a) Packing in the first $\lfloor k/2 \rfloor$ bins

(b) Packing in the last $\lceil k/2 \rceil$ bins

Figure 19: Packing of the bins in the proof of Lemma 14

the values for $r_A^{(i)}, r_B^{(i)}$ and $r_C^{(i)}$ by exchanging the widths and the heights. Consequently, $w_D^{(j)} \leq 12/24 - \varepsilon_c, h_D^{(j)} \leq 14/24, w_E^{(j)} \leq 4/24, h_E^{(j)} = 1, w_F^{(j)} = x_b^{(j)} \leq 1 - x_u'^{(j)}$ and $h_F^{(j)} \leq 12/24 - \varepsilon_c$.



(a) Packing in the additional bin C_1

(b) Packing in the additional bin C_2

Figure 20: Packing of the additional bins in the proof of Lemma 14

In a first step, we move the regions out of the bins. Analogously to Lemma 12, we place $r_C^{(i)}$ on the position $(1/2, y_\ell'^{(i)})$ and $r_F^{(j)}$ on the position $(x_u'^{(j)}, 1/2)$ (cf. Figure 19). Let there be two bins B_1, B_2 of the first $\lfloor k/2 \rfloor$ bins and two bins B_3, B_4 of the remaining bins. We move the rectangles $r_A^{(1)}, r_A^{(2)}$ and $r_E^{(3)}, r_E^{(4)}$ into an additional bin C_1 and the rectangles $r_D^{(3)}, r_D^{(4)}$ and $r_B^{(1)}, r_B^{(2)}$ into an additional bin C_2 (cf. Figure 20).

We pack $r_A^{(1)}$ in the lower left corner of bin C_1 on the position $(0, 0)$ and $r_A^{(2)}$ on top of it on the position $(0, 1/2)$. The widths of both rectangles are at most $14/24$. On the right side there is enough space to place $r_E^{(3)}$ and $r_E^{(4)}$. These rectangles have a total width of $8/24$. On the right side there is still a free space of width $2/24$ and hence the strip S_R is completely free of rectangles.

The packing in C_2 is analogous. $r_D^{(3)}$ and $r_D^{(4)}$ are placed next to each other at the bottom of bin C_2 on the positions $(0, 0)$ and $(1/2, 0)$. On top of them there is a free space of at least $10/24$. We place $r_B^{(1)}$ and $r_B^{(2)}$ on top of them, leaving the strip S_U free of rectangles. We repeat this method with each sequence of 4 of the k bins. Having observed the same results as in the last paragraph of the proof of Lemma 10, we need $3/2k + 1$ bins in total, when k is not a multiple of 4. In each bin there is either a horizontal or a vertical strip free of rectangles, hence we are able to employ Lemma 3 or Lemma 4 to satisfy either Property 1 or Property 2. □

The next lemma considers the case that the horizontal intervals are close to the bottom and to the top of the bin while the vertical intervals are close to the middle.

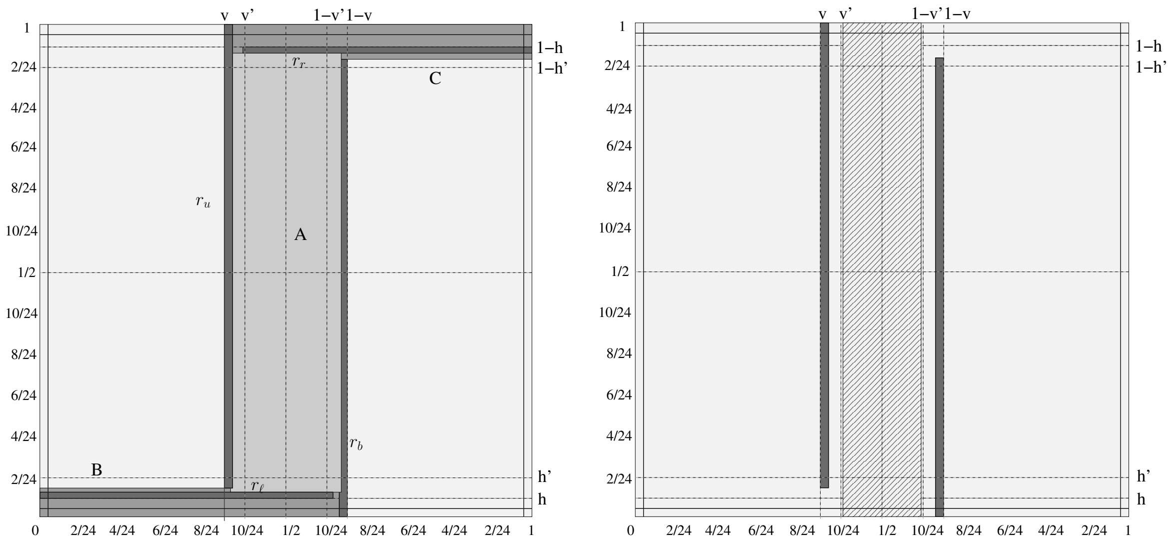
Lemma 15. *Let $B_1, \ldots, B_k$ be k bins so that each bin B_i, for $i \in \{1, \ldots, k\}$, has a packing with the following conditions:*

2.18. $v^{(i)} \in \{8/24, 9/24, 10/24, 11/24\}$, *2.19.* $h^{(i)} \in \{0/24, 1/24\}$.

It follows that we are able to round up the rectangles in these bins and rearrange them into $3/2 \cdot k + 2$ *bins, while the packing of each of them satisfies either Property 1 or Property 2.*

Proof. Let $i \in \{1, \ldots, k\}$. We use a similar region definition as in the lemmas before. The height of the region in the middle is very large, but the regions at the top and at the bottom are small.

Let $A^{(i)}$ be the rectangular region defined by the points $(x_u'^{(i)}, y_\ell'^{(i)}), (x_b^{(i)}, y_\ell'^{(i)}), (x_b^{(i)}, y_r^{(i)})$ and $(x_u'^{(i)}, y_r^{(i)})$. The corresponding rectangle $r_A^{(i)}$ of this region has a width of $w_A^{(i)} = x_b^{(i)} - x_u'^{(i)} \leq 1 - v^{(i)} - v^{(i)} \leq 16/24 - 8/24 = 1/3$ (Condition 2.18). The rectangles $r_\ell^{(i)}$ and $r_r^{(i)}$ are not completely situated inside the strips $S_B^{(i)}$ and $S_U^{(i)}$, as they would have been removed in Lemma 1. It follows that the height of $r_A^{(i)}$ is bounded by $h_A^{(i)} = y_r^{(i)} - y_\ell'^{(i)} \leq 1 - 2\varepsilon_c \leq 1 - \varepsilon_c$. In the next step, we define the region $B^{(i)}$ by the coordinates $(0, 0), (x_b^{(i)}, 0), (x_b^{(i)}, y_u^{(i)}), (0, y_u^{(i)})$

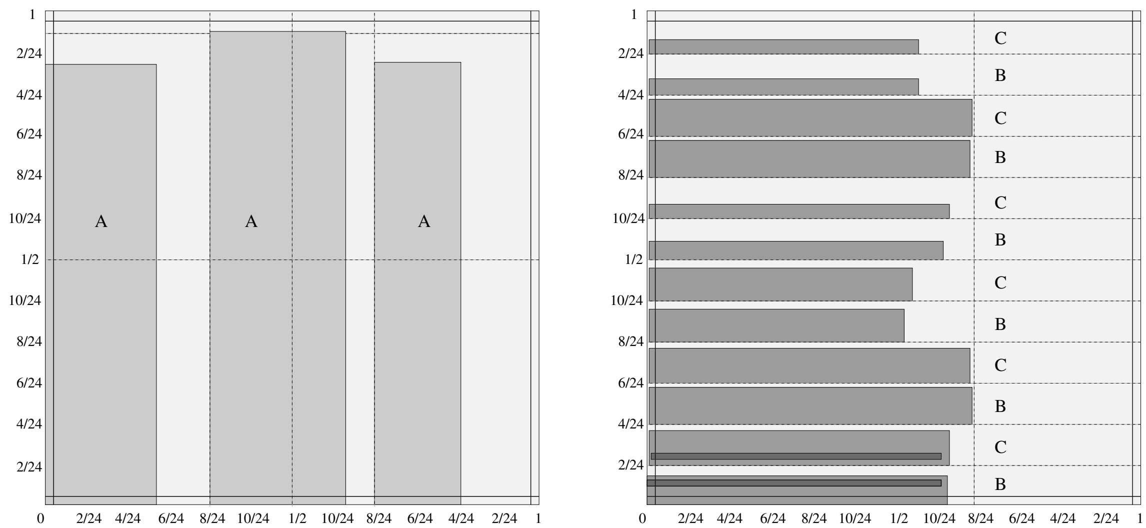


(a) Definition of the regions $A^{(i)}$,$B^{(i)}$ and $C^{(i)}$ (b) The whole strip in the middle is free of rectangles

Figure 21: Definitions of the regions in the proof of Lemma 15

and the region $C^{(i)}$ by the coordinates $(x_u'^{(i)}, y_b'^{(i)}), (1, y_b'^{(i)}), (1, 1), (x_u'^{(i)}, 1)$. The heights of the corresponding containers/rectangles $r_B^{(i)}$ and $r_C^{(i)}$ of these regions are bounded by $h_B^{(i)} = y_u^{(i)} - 0 \leq h'^{(i)} \leq 2/24$ and $h_C^{(i)} = 1 - y_b'^{(i)} \leq 1 - (1 - h'^{(i)}) = h'^{(i)} \leq 2/24$ (Condition 2.19). The widths are bounded by $w_B^{(i)} = x_b^{(i)} - 0 \leq 1 - v^{(i)} \leq 1 - 8/24 = 16/24 = 2/3$ and $w_C^{(i)} = 1 - x_u'^{(i)} \leq 1 - v^{(i)} \leq 1 - 8/24 = 16/24 = 2/3$ (Condition 2.18). We move these rectangles and with them all rectangles that are completely situated inside these regions into additional bins. There are some rectangles left that intersect the region $A^{(i)}$ and that are not completely situated in one of these regions. These rectangles have to intersect $A^{(i)}$ from above or below since all rectangles that intersect $A^{(i)}$ from the left are below $r_u^{(i)}$ and hence located inside $B^{(i)}$ and the rectangles that intersect from the right have to lie above $r_b^{(i)}$ and are hence situated inside $C^{(i)}$. The rectangles that intersect $A^{(i)}$ from below have to be located between $r_\ell^{(i)}$ and $r_b^{(i)}$; the rectangles that intersect from above have to be situated between $r_u^{(i)}$ and $r_r^{(i)}$. It follows that, after moving the rectangles $r_A^{(i)}, r_B^{(i)}$ and $r_C^{(i)}$ into additional bins, the complete vertical strip between $x_r^{(i)}$ and $x_\ell'^{(i)}$ is completely free of rectangles (cf. Figure 21). Since $x_\ell'^{(i)} \geq 1 - v'^{(i)} \geq 1 - (12/24 - \varepsilon_c) = 1/2 + \varepsilon_c$ and $x_r^{(i)} \leq v'^{(i)} \leq 12/24 - \varepsilon_c = 1/2 - \varepsilon_c$, the strip has a width of at least $2\varepsilon_c$. Thus, we can move all rectangles on the right of the x'-coordinate $x_\ell'^{(i)}$ by ε_c to the left and secure that the strip $S_R^{(i)}$ is completely free of rectangles.

We move the rectangles of each sequence of 6 bins $B_1, \ldots, B_6$ of the k bins into 3 additional bins C_1, C_2, C_3. The rectangles $r_A^{(1)}, r_A^{(2)}$ and $r_A^{(3)}$ are moved into bin C_1, the rectangles $r_A^{(4)}, r_A^{(5)}, r_A^{(6)}$ are moved into bin C_2 and the remaining rectangles into bin C_3. Each rectangle $r_A^{(i)}$ has a width of at most $1/3$. Thus, we place them on the positions $(0,0), (1/3,0)$ and $(2/3,0)$ into bins C_1 and C_2. The uppermost horizontal strip S_U of height ε_c is still free of rectangles. The total height of the rectangles $r_B^{(1)}, \ldots, r_B^{(6)}$ and the rectangles $r_C^{(1)}, \ldots, r_C^{(6)}$ is $6 \cdot (2/24 + 2/24) = 1$. We place them on top of each other into bin C_3 with their y-coordinates situated on position 0. Since they have a width of at most $2/3$ the strip S_R is still free (cf. Figure 22).



(a) Packing in the bin C_1 and C_2

(b) Packing in the bin C_3

Figure 22: Packing in the additional bins in the proof of Lemma 15

Accordingly to the discussion in the last paragraph of the proof of Lemma 8, we need $3/2k + 2$ bins in total, if k is not a multiple of 6. In each bin there is either a vertical strip of the width ε_c or a horizontal strip of the height ε_c free of rectangles. This allows us to employ Lemma 3 or Lemma 4 in order to round up the rectangles and to satisfy either Property 1 or Property 2. $\square$

The analogous lemma is stated as follows. In the version that allows rotation, we only use Lemma 15.

Lemma 16. *Let $B_1, \ldots, B_k$ be k bins so that each bin B_i, for $i \in \{1, \ldots, k\}$, has a packing with the following conditions:*

2.20. $h^{(i)} \in \{8/24, 9/24, 10/24, 11/24\}$,

2.21. $v^{(i)} \in \{0/24, 1/24\}$.

It follows that we are able to round up the rectangles in these bins and rearrange them into $3/2 \cdot k + 2$ bins, while the packing of each of them satisfies either Property 1 or Property 2.

Before concluding our case analysis, there is one remaining case left in which either $h^{(i)} = 12/24 - \varepsilon_c$ or $v^{(i)} = 12/24 - \varepsilon_c$.

2.2.2 Last Remaining Case

We distinguish, if the rectangles in the corners intersect the intervals that are very close to the middle of the bin, or if they do not. If $v^{(i)} = 12/24 - \varepsilon_c$ and $r_{u\ell}^{(i)} \in L_U^{(i)}$, then the width of $r_{u\ell}^{(i)}$ is close to $1/2$ and the height is larger than $1/2$. If $r_{u\ell}^{(i)} \notin L_U^{(i)}$, then there are only rectangles of a total width of ε_c in $I_L^{(i)}$. In the first lemma, we have $v^{(i)} = 12/24 - \varepsilon_c$ and $r_{u\ell}^{(i)} \in L_U^{(i)}$ and $r_{br}^{(i)} \in L_B^{(i)}$. Hence there are two very large rectangles in the packing.

Lemma 17. *Let $B_1, \dots, B_k$ be k bins so that each bin B_i, for $i \in \{1, \dots, k\}$, has a packing with the following conditions:*

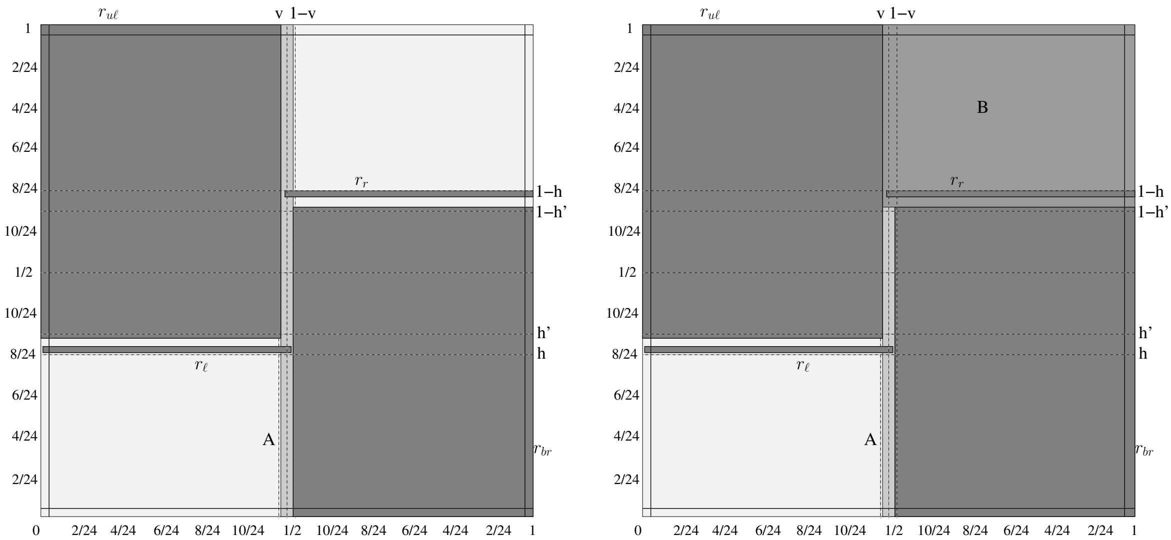
2.22. $v^{(i)} = 12/24 - \varepsilon_c$,

2.23. $h^{(i)} \in \mathrm{IN}$,

2.24. $r_{u\ell}^{(i)} \in L_U^{(i)}$ *and* $r_{br}^{(i)} \in L_B^{(i)}$.

It follows that we are able to round up the rectangles in these bins and rearrange them into $3/2 \cdot k + 1$ bins, while the packing of each of them satisfies Property 1.

Proof. Let $i \in \{1, \dots, k\}$. The rectangles $r_{u\ell}^{(i)}$ and $r_{br}^{(i)}$ have a width of at least $1/2 - \varepsilon_c$, since they have to intersect the interval $\mathrm{VL}_{12/24-\varepsilon_c}^{(i)}$ and $\mathrm{VR}_{12/24-\varepsilon_c}^{(i)}$. Therefore, the space between the rectangles is very small. W.l.o.g. let $h_{u\ell}^{(i)} \geq h_{br}^{(i)}$ since we are able to turn the bin by 180°. We define the region $A^{(i)}$ between these rectangles by the coordinates $(x_{u\ell}'^{(i)}, 0)$, $(x_{br}^{(i)}, 0)$,



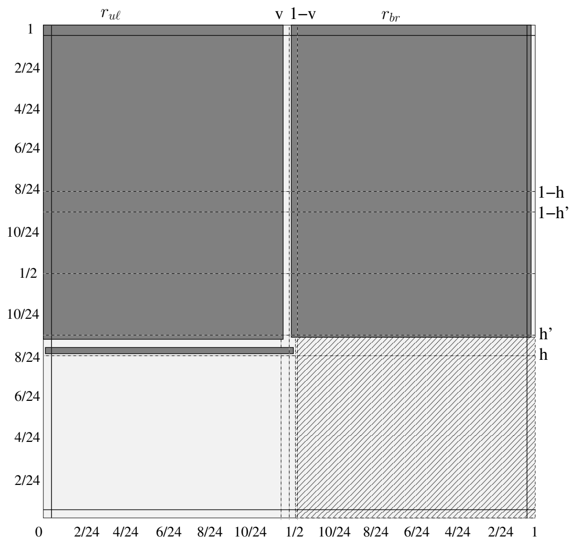
(a) Definition of the region $A^{(i)}$

(b) Definition of the region $B^{(i)}$

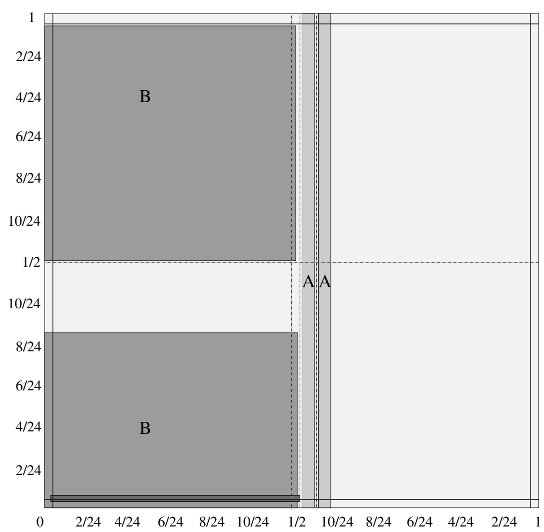
Figure 23: Definitions of the regions in the proof of Lemma 17

$(x_{br}^{(i)}, 1)$ and $(x_{u\ell}'^{(i)}, 1)$. We treat this region as a rectangle $r_A^{(i)}$ of height $h_A^{(i)} = 1$ and width

$w_A^{(i)} = x_{br}^{(i)} - x_{u\ell}^{\prime(i)} \leq (1-v^{(i)}) - v^{(i)} = 1-(12/24-\varepsilon_c)-(12/24-\varepsilon_c) = 2\varepsilon_c$ (Condition 2.22). Moreover, we define a region $B^{(i)}$ by the coordinates $(x_{u\ell}^{\prime(i)}, y_{br}^{\prime(i)})$, $(1, y_{br}^{\prime(i)})$, $(1,1)$ and $(x_{u\ell}^{\prime(i)}, 1)$. The corresponding rectangle $r_B^{(i)}$ has a width of at most $w_B^{(i)} = 1-x_{u\ell}^{\prime(i)} \leq 1-v^{(i)} = 1-(12/24-\varepsilon_c) = 12/24+\varepsilon_c = 1/2+\varepsilon_c$ and a height of at most $h_B^{(i)} = 1-y_{br}^{\prime(i)} \leq 1-(1-h^{\prime(i)}) \leq 1/2$ (cf. Figure 23). We move the rectangles $r_A^{(i)}$ and $r_B^{(i)}$ of each sequence of 2 bins into one additional bin. We move $r_{br}^{(i)}$ out of bin B_i for a moment. There is no rectangle on the right of $r_{u\ell}^{(i)}$, since all



(a) Packing in the bin B_i



(b) Packing in the bin C_1

Figure 24: Packing in the proof of Lemma 17

rectangles that were on top of $r_{br}^{(i)}$ are situated in $r_B^{(i)}$ and all rectangles that are located between $r_{u\ell}^{(i)}$ and $r_{br}^{(i)}$ are now to be found in $r_A^{(i)}$. At this moment, we use Lemma 3 on this bin since $S_R^{(i)}$ is completely free of rectangles. The width of the rectangle $r_{u\ell}^{(i)}$ is rounded up to the next largest multiple of $\varepsilon_c^2/2$ which is at most $1/2$. Thus, the new x'-coordinate is $x_{u\ell}^{\prime(i)} \leq 1/2$. We reinsert the rectangle $r_{br}^{(i)}$ into the bin after rounding up its width to the next largest multiple of $\varepsilon_c^2/2$ to at most $1/2$. Since $h_{u\ell}^{(i)} \geq h_{br}^{(i)}$, we can place $r_{br}^{(i)}$ on the right side of $r_{u\ell}^{(i)}$ on the x-coordinate $1/2$ (cf. Figure 24).

Let there be two bins B_1, B_2 of the k bins and let C_1 be an additional bin. We pack $r_B^{(1)}$ and $r_B^{(2)}$ on top of each other on the positions $(0,0)$ and $(0,1/2)$ into C_1. On the right side there is still a free space of at least $12/24-\varepsilon_c$. Hence, there is enough space to place $r_A^{(1)}$ on the position $(12/24+\varepsilon_c, 0)$ and $r_A^{(2)}$ on the position $(12/24+3\varepsilon_c, 0)$. There is still a space of at least $12/24-5\varepsilon_c$ free of rectangles including the strip S_R. Thus, we are able to employ Lemma 3 on bin C_1. In total, we have $3/2k+1$ bins when k is odd while the packing of each bin satisfies Property 1. □

By exchanging the values $v^{(i)}$ and $h^{(i)}$, we obtain the following lemma. Since it is the analogous lemma, by turning the bin by 90°, we do not need it in the version that allows rotation.

Lemma 18. *Let $B_1, \dots, B_k$ be k bins so that each bin B_i, for $i \in \{1, \dots, k\}$, has a packing with the following conditions:*

2.25. $v^{(i)} \in \text{IN}$,
2.26. $h^{(i)} = 12/24 - \varepsilon_c$,
2.27. $r_{ur}^{(i)} \in W_R^{(i)}$ *and* $r_{b\ell}^{(i)} \in W_L^{(i)}$.

It follows that we are able to round up the rectangles in these bins and rearrange them into $3/2 \cdot k + 1$ *bins, while the packing of each of them satisfies Property 2.*

If there are no such big rectangles in the packing of bin B_i, we cannot adopt the method described above. However, when $v^{(i)} = 12/24 - \varepsilon_c$, the total width of either $L_U^{(i)}$ or $L_B^{(i)}$ is very small.

Lemma 19. *Let* $B_1, \ldots, B_k$ *be* k *bins so that each bin* B_i*, for* $i \in \{1, \ldots, k\}$*, has a packing with the following conditions:*

2.28. $v^{(i)} = 12/24 - \varepsilon_c$,
2.29. $h^{(i)} \in \text{IN}$,
2.30. $r_{u\ell}^{(i)} \notin L_U^{(i)}$ *or* $r_{br}^{(i)} \notin L_B^{(i)}$.

It follows that we are able to round up the rectangles in these bins and rearrange them into $(3/2 + \varepsilon_c) \cdot k + 2$ *bins, while the packing of each of them satisfies Property 2.*

Proof. Let $i \in \{1, \ldots, k\}$. W.l.o.g. we suppose that $r_{u\ell}^{(i)} \notin L_U^{(i)}$ since we are able to turn the bin by $180°$. The rectangles in $L_U^{(i)}$ have a total width of at most ε_c since they have to be completely situated in the interval $\text{VL}_{12/24-\varepsilon_c}^{(i)}$. We move these rectangles in $L_U^{(i)}$ into an additional bin. We are able to pack $1/\varepsilon_c$ sets next to each other at the bottom. Therefore, we need at most $\lceil \varepsilon_c \cdot k \rceil < \varepsilon_c \cdot k + 1$ additional bins for all k bins. We are able to round up the heights of these rectangles to the next multiple of $\varepsilon_c^2/2$ to at most 1 and satisfy Property 2.

No rectangle of a height larger than $1/2$ remains to intersect $S_U^{(i)}$. Thus, we are able to employ Lemma 6 with $y = 0$ and $x = 1/2$, and achieve a packing into $3/2 \cdot k + 1$ bins, while each packing satisfies Property 2. In total, we have a packing of $(3/2 + \varepsilon_c) \cdot k + 2$ bins. □

The last remaining case is analogous to Lemma 19. We do not need it in the version that allows rotation.

Lemma 20. *Let* $B_1, \ldots, B_k$ *be* k *bins so that each bin* B_i*, for* $i \in \{1, \ldots, k\}$*, has a packing with the following conditions:*

2.31. $v^{(i)} \in \text{IN}$,
2.32. $h^{(i)} = 12/24 - \varepsilon_c$,
2.33. $r_{ur}^{(i)} \notin W_R^{(i)}$ *or* $r_{b\ell}^{(i)} \notin W_L^{(i)}$.

It follows that we are able to round up the rectangles in these bins and rearrange them into $(3/2 + \varepsilon_c) \cdot k + 2$ *bins, while the packing of each of them satisfies Property 1.*

2.2.3 Résumé of the Case Analysis

In conclusion we obtain the following theorem:

Theorem 4. *For any value* ε_c*, with* $1/\varepsilon_c$ *being a multiple of* 24*, and for any solution that fits into* m *bins, we are able to round up the widths and the heights of the rectangles so that they fit into* $(3/2 + 5\varepsilon_c) \cdot m + 37$ *bins while the packing of each of the bins satisfies either Property 1 or Property 2.*

Proof. Let B_i be a bin in our solution. We prove that no matter how the packing looks like, one of the previous lemmas can be employed. A packing that does not satisfy the properties of Theorem 3 can be solved with Lemma 2 or Lemma 7. Thus, we suppose that the packing has the properties of Theorem 3 with certain values $h^{(i)} \in \mathrm{IN}$ and $v^{(i)} \in \mathrm{IN}$. Depending on these values and whether the rectangles in the corner have width or height larger than $1/2$, we use different lemmas (cf. Figure 25).

Case 1: $h^{(i)}$ and $v^{(i)}$ are both in the set $\{0/24, \ldots, 8/24 - \varepsilon_c\}$.
If $r_{b\ell}^{(i)} \notin W_L^{(i)}$ or $r_{ur}^{(i)} \notin W_R^{(i)}$, we employ Lemma 8, if $r_{u\ell}^{(i)} \notin L_U^{(i)}$ or $r_{br}^{(i)} \notin L_B^{(i)}$, we employ Lemma 9. Consequently, we conclude $r_{b\ell}^{(i)} \in W_L^{(i)}$, $r_{ur}^{(i)} \in W_R^{(i)}$, $r_{u\ell}^{(i)} \in L_U^{(i)}$ and $r_{br}^{(i)} \in L_B^{(i)}$. We solve this case either according to Lemma 10 or, if $h^{(i)} = 8/24 - \varepsilon_c$, according to Lemma 11.

Case 2: $v^{(i)} = 12/24 - \varepsilon_c$ or $h^{(i)} = 12/24 - \varepsilon_c$.
If $v^{(i)} = 12/24 - \varepsilon_c$, we employ either Lemma 17, if $r_{u\ell}^{(i)} \in L_U^{(i)}$ and $r_{br}^{(i)} \in L_B^{(i)}$ and else Lemma 19. If $h^{(i)} = 12/24 - \varepsilon_c$ and if $r_{ur}^{(i)} \in W_R^{(i)}$ and $r_{b\ell}^{(i)} \in W_L^{(i)}$ we adopt Lemma 18 and else Lemma 20.

Case 3: $v^{(i)} \in \{8/24, \ldots, 11/24\}$ and $h^{(i)} \in \{0/24, \ldots, 9/24\}$.
If $h^{(i)} \in \{0/24, 1/24\}$ we make use of Lemma 15. Otherwise, if $h^{(i)} \in \{2/24, \ldots, 9/24\}$, we employ Lemma 12.

Case 4: $h^{(i)} \in \{8/24, \ldots, 11/24\}$ and $v^{(i)} \in \{0/24, \ldots, 9/24\}$.
Analogous to the third case we use Lemma 16 if $v^{(i)} \in \{0/24, 1/24\}$ and Lemma 13 if $v^{(i)} \in \{2/24, \ldots, 9/24\}$.

Case 5: $h^{(i)}$ and $v^{(i)}$ are both in the set $\{10/24, 11/24\}$.
In this case we employ Lemma 14.

In each lemma mentioned here we modify a packing of k bins into a packing with at most $(3/2 + \varepsilon_c)k$ bins and a constant number of additional bins. Furthermore, we remove all rectangles that are completely in the strips at the sides of the bin and need therefore $4\varepsilon_c \cdot m + 2$ additional bins in Lemma 1. It follows that we need $(3/2 + 5\varepsilon_c) \cdot m$ bins plus a constant number of bins in total.

In Lemma 7, the constant number of additional bins is 15. Two additional bins are needed in the Lemma 8, Lemma 9,Lemma 11,Lemma 15,Lemma 16, Lemma 19 and Lemma 20. We need one additional bin in the Lemma 10, Lemma 12, Lemma 13, Lemma 14, Lemma 17 and Lemma 18. By summing up these constant numbers, we obtain a value of $2 + 15 + 2 \cdot 7 + 6 = 2 + 15 + 20 = 37$.

$v^{(i)} \backslash h^{(i)}$	0/24	1/24	2/24	3/24	4/24	5/24	6/24	7/24	$8/24 - \varepsilon_c$	8/24	9/24	10/24	11/24	$12/24 - \varepsilon_c$
0/24	8,9,10	8,9,10	8,9,10	8,9,10	8,9,10	8,9,10	8,9,10	8,9,10	8,9,11	16	16	16	16	18,20
1/24	8,9,10	8,9,10	8,9,10	8,9,10	8,9,10	8,9,10	8,9,10	8,9,10	8,9,11	16	16	16	16	18,20
2/24	8,9,10	8,9,10	8,9,10	8,9,10	8,9,10	8,9,10	8,9,10	8,9,10	8,9,11	13	13	13	13	18,20
3/24	8,9,10	8,9,10	8,9,10	8,9,10	8,9,10	8,9,10	8,9,10	8,9,10	8,9,11	13	13	13	13	18,20
4/24	8,9,10	8,9,10	8,9,10	8,9,10	8,9,10	8,9,10	8,9,10	8,9,10	8,9,11	13	13	13	13	18,20
5/24	8,9,10	8,9,10	8,9,10	8,9,10	8,9,10	8,9,10	8,9,10	8,9,10	8,9,11	13	13	13	13	18,20
6/24	8,9,10	8,9,10	8,9,10	8,9,10	8,9,10	8,9,10	8,9,10	8,9,10	8,9,11	13	13	13	13	18,20
7/24	8,9,10	8,9,10	8,9,10	8,9,10	8,9,10	8,9,10	8,9,10	8,9,10	8,9,11	13	13	13	13	18,20
$8/24 - \varepsilon_c$	8,9,10	8,9,10	8,9,10	8,9,10	8,9,10	8,9,10	8,9,10	8,9,10	8,9,11	13	13	13	13	18,20
8/24	15	15	12	12	12	12	12	12	12	12,13	12,13	13	13	18,20
9/24	15	15	12	12	12	12	12	12	12	12,13	12,13	13	13	18,20
10/24	15	15	12	12	12	12	12	12	12	12	12	14	14	18,20
11/24	15	15	12	12	12	12	12	12	12	12	12	14	14	18,20
$12/24 - \varepsilon_c$	17,19	17,19	17,19	17,19	17,19	17,19	17,19	17,19	17,19	17,19	17,19	17,19	17,19	17,18,19,20

Figure 25: Overview of the lemmas that are applied for different v and h

□

Résumé of the Case Analysis with Rotations The additive constant changes in the version that allows rotation. We employ Lemma' 1 with an additive constant of 1 and Lemma' 7 with an additive constant of 8 instead of Lemma 1 with an additive constant of 2 and Lemma 7 with an additive constant of 15, respectively. Furthermore, we do not need Lemma 9, Lemma 13, Lemma 16, Lemma 18 and Lemma 20. This reduces the additive constant to the value $1 + 8 + 2 \cdot 4 + 4 = 21$.

Theorem' 4. *For any value ε_c, with $1/\varepsilon_c$ being a multiple of 24, and for any solution that fits into m bins, we are able to rotate and to round up the widths of the rectangles so that they fit into $(3/2 + 5\varepsilon_c) \cdot m + 21$ bins while the packing of each of the bins satisfies Property 1.*

2.3 Rounding the Other Side

In this section, we round the remaining unrounded side of the rectangles after employing Theorem 4 on an optimal solution in OPT bins. However, before we adopt Theorem 4 we divide the instance into big, wide, long, small and medium rectangles. Let $\varepsilon' \leq \min\{\varepsilon/39, 1/48\}$, so that $1/\varepsilon'$ is a multiple of 24. Similar as in [16], we find a value δ, so that the rectangles with at least one side length between δ and δ^4 have a small total area.

Lemma 21. *We find a value $\delta \leq \varepsilon'$, so that $1/\delta$ is a multiple of 24 and all rectangles r_i of the width $w_i \in [\delta^4, \delta)$ or the height $h_i \in [\delta^4, \delta]$ have a total area of at most $\varepsilon' \cdot \mathrm{OPT}$.*

Proof. Define a sequence $\sigma_1, \ldots, \sigma_{2/\varepsilon'+1}$, whereas σ_1 is the largest value with $\sigma_1 = \varepsilon'$ and $\sigma_{k+1} = \sigma_k^4$, for $k \in \{1, \ldots, 2/\varepsilon'\}$. Each reciprocal of the members in the sequence is a multiple of 24. This is a consequence of an inductive argument: $1/\varepsilon'$ is a multiple of 24; let $1/\sigma_k = i \cdot 24$ for one integer i. It follows that, $1/\sigma_{k+1} = 1/\sigma_k^4 = (1/\sigma_k)^4 = (i \cdot 24)^4 = (i^4 \cdot 24^3) \cdot 24$. Hence, $1/\sigma_{k+1}$ is a multiple of 24.

Let M_{σ_k} be the set of rectangles r_i with $w_i \in [\sigma_{k+1}, \sigma_k)$ or $h_i \in [\sigma_{k+1}, \sigma_k)$. Each rectangle r_i in the instance belongs to at most two sets M_{σ_k}. Since the area is a lower bound for OPT we have $\sum_{k=1}^{2/\varepsilon'} a(M_{\sigma_k}) \leq 2\mathrm{OPT}$. Suppose that all sets have an area of $a(M_{\sigma_k}) > \varepsilon' \cdot \mathrm{OPT}$, for all $k \in \{1, \ldots, 2/\varepsilon\}$. We obtain $\sum_{k=1}^{2/\varepsilon'} a(M_{\sigma_k}) > 2/\varepsilon' \cdot \varepsilon'\mathrm{OPT} = 2 \cdot \mathrm{OPT}$, which is a contradiction. Therefore, there exists at least one set with $a(M_{\sigma_k}) \leq \varepsilon' \cdot \mathrm{OPT}$. We set $\delta := \sigma_k$, so that k is the smallest value with $a(M_{\sigma_k}) \leq \varepsilon' \cdot \mathrm{OPT}$. □

The rectangles are separated into *big* rectangles of width and height at least δ, *wide* rectangles of width at least δ and height smaller than δ^4, *long* rectangles of width smaller than δ^4 and height at least δ, *small* rectangles of width and height less than δ^4 and *medium* rectangles of width or height in $[\delta^4, \delta)$.

Since $1/\delta$ is a multiple of 24 we are able to adopt Theorem 4 with an optimal solution consisting of OPT bins and with $\varepsilon_c := \delta$. The resulting solution consists of $k \leq (3/2 + 5\delta)\mathrm{OPT} + 37$ bins which satisfy Property 1 or Property 2. A bin is of Type 1, if its packing satisfies Property 1 and otherwise it is of Type 2. Therefore, let $B_1, \ldots, B_{p_1}$, be the bins of Type 1 and $B_{p_1+1}, \ldots, B_k$ be the bins of Type 2.

The widths of the big and wide rectangles in the bins $B_1, \ldots, B_{p_1}$ and the heights of the big and long rectangles in the bins $B_{p_1+1}, \ldots, B_k$ are therefore multiples of $\delta^2/2$. We denote the set of big and wide rectangles that are packed into the bins $B_1, \ldots, B_{p_1}$ and that have a width of $i\delta^2/2$, for $i \in \{2/\delta, \ldots, 2/\delta^2\}$ by B_i^w and W_i^w, respectively. The set of big and long rectangles that are packed in the bins $B_{p_1+1}, \ldots, B_k$ and that have a height of $i\delta^2/2$ are denoted by B_i^h and L_i^h, respectively. The wide rectangles that are packed in the bins $B_{p_1+1}, \ldots, B_k$ are denoted by

W^h and the long rectangles that are packed in the bins $B_1, \dots, B_{p_1}$ are denoted by L^w. The set of small rectangles is denoted by S. The set of medium rectangles that have a width within $[\delta^4, \delta)$ are denoted by $M_{w\delta}$ and the remaining rectangles of height within $[\delta^4, \delta)$ are denoted by $M_{h\delta}$.

The medium rectangles have a total area of at most $\varepsilon' \cdot \mathrm{OPT}$. Therefore, we are able to move them into few additional bins with Steinberg's Algorithm [21].

Theorem 5 (Steinberg's algorithm). *If the following inequalities hold,*

$$w_{\max}(T) \leq a, \;\; h_{\max}(T) \leq b, \;\; \textit{and} \quad 2a(T) \leq ab - (2w_{\max}(T) - a)_+(2h_{\max}(T) - b)_+$$

where $x_+ = \max(x, 0)$, then it is possible to pack all items from T into $R = (a, b)$ in time $\mathcal{O}((n \log^2 n)/ \log \log n)$.

Therefore, we only have to partition the set of medium rectangles into subsets of a total area of at most $1/2$.

Lemma 22. *We pack the medium rectangles into at most $3\varepsilon'\mathrm{OPT} + 2$ additional bins*

Proof. We split the sets $M_{h\delta}$ and $M_{w\delta}$ into sets of a total area of at most $1/2$. Each medium rectangle has an area of at most δ. Therefore, we are able to greedily divide $M_{h\delta}$ and $M_{w\delta}$ into sets of rectangles of a total area within $(1/2 - \delta, 1/2]$ and two additional sets of rectangles with a bounded total area of at most $1/2 - \delta$. It holds that $3\varepsilon'\mathrm{OPT} \cdot (1/2 - \delta) > 3\varepsilon'\mathrm{OPT} \cdot (1/2 - 1/6) = \varepsilon'\mathrm{OPT}$, since $\delta \leq \varepsilon' \leq 1/48 < 1/6$. Hence, the total number of sets is bounded by $\lceil 3\varepsilon'\mathrm{OPT} \rceil \leq 3\varepsilon'\mathrm{OPT} + 2$. The rectangles in each set have either a maximum width of at most $\delta < 1/2$ or a maximum height of at most $\delta < 1/2$. This enables us to pack each set into one bin using Steinberg's Theorem 5. ☐

In the next step, we round up the heights of the big and long rectangles in the bins $B_1, \dots, B_{p_1}$ and the widths of the big and wide rectangles in the bins $B_{p_1+1}, \dots, B_k$. Furthermore, we pack the wide and long rectangles fractionally into wide and long containers.

Packing Medium Rectangles with Rotations We use the same construction in the version that allows rotation. The difference is that we rotate the rectangles in the set $M_{h\delta}$. Consequently, these rectangles belong to the set $M_{w\delta}$. We partition this set analogous as in the proof of Lemma 22 into $3\varepsilon'\mathrm{OPT} + 1$ subsets of total area at most $1/2$. These sets are packed with Steinberg's Theorem 5. We obtain the following lemma for the version that allows rotation.

Lemma' 22. *We rotate and pack the medium rectangles into at most $3\varepsilon'\mathrm{OPT} + 1$ additional bins*

The next steps are explained for the bins $B_1, \dots, B_{p_1}$, the rounding for the remaining bins is analogous.

2.3.1 Rounding Big and Long Rectangles

We round the heights of the big and long rectangles in the sets B_i^w and L_i^w, for each $i \in \{2/\delta, \dots, 2/\delta^2\}$.

To do this, we adopt a similar rounding technique as in the algorithm by Kenyon & Rémila [19] and in the algorithm by Fernandez de la Vega & Lueker [10]. We focus on one set B_i^w of big rectangles, for $i \in \{2/\delta, \dots, 2/\delta^2\}$. We sort the rectangles in this set according to non-decreasing heights. Let k_i be the number of rectangles in B_i^w, denoted by $r_{i,1}, \dots, r_{i,k_i}$. The rectangle $r_{i,1}$ has the largest and r_{i,k_i} the smallest height. We define at most $1/\delta^2$ subsets $B^w_{i,j}$, which consist of $\lfloor \delta^2 \cdot k_i \rfloor$ rectangles except the last subset with possibly less items. This is done by assigning $\lfloor \delta^2 \cdot k_i \rfloor$ rectangles into one subset, then we leave one rectangle out and assign the next $\lfloor \delta^2 \cdot k_i \rfloor$ rectangles into the next subset. Thus, the first rectangle $r_{i,1}$ is not assigned to a subset and is called the first cut-rectangle. The rectangles $r_{i,2}, \ldots, r_{i,\lfloor \delta^2 \cdot k_i \rfloor+1}$ are assigned to subset $B^w_{i,1}$. The rectangle $r_{i,\lfloor \delta^2 \cdot k_i \rfloor+2}$ is called the second cut-rectangle. The rectangles $r_{i,\lfloor \delta^2 \cdot k_i \rfloor+3}, \ldots, r_{i,2\cdot\lfloor \delta^2 \cdot k_i \rfloor+2}$ are assigned to subset $B^w_{i,2}$ and so on. Hence, the jth cut-rectangle is $r_{i,(j-1)\cdot\lfloor \delta^2 \cdot k_i \rfloor+j}$ and the subset $B^w_{i,j}$ contains the rectangles $r_{i,(j-1)\cdot\lfloor \delta^2 \cdot k_i \rfloor+(j+1)}, \ldots, r_{i,j\cdot\lfloor \delta^2 \cdot k_i \rfloor+j}$. We have at most $1/\delta^2$ cut-rectangles and subsets, since we have at most $1/\delta^2 \cdot (1 + \lfloor \delta^2 \cdot k_i \rfloor) \geq 1/\delta^2 \cdot \delta^2 \cdot k_i = k_i$ rectangles.

We round the heights of the rectangles in each subset $B^w_{i,j}$ to the height of the jth cut-rectangle. Afterwards, we move the rectangles of the first subset $B^w_{i,1}$ into additional bins. Note that the total width of subset $B^w_{i,1}$, denoted by $w(B^w_{i,1})$, is at most $\lfloor \delta^2 \cdot k_i \rfloor \cdot i\delta^2/2 \leq \delta^2 \cdot k_i \cdot i\delta^2/2 = \delta^2 \cdot w(B^w_i)$, with $w(B^w_i)$ being the total width of all rectangles in B^w_i. Each rectangle in a remaining subset $B^w_{i,j}$ is placed on a position of one rectangle in subset $B^w_{i,j-1}$. This is done by placing the ℓth rectangle of subset $B^w_{i,j}$ on the position of the ℓth rectangle of subset $B^w_{i,j-1}$. This is possible since all rectangles have the same width, all subsets, except the last, have the same cardinality and the height of the ℓth rectangle in subset $B^w_{i,j-1}$ is larger than or equal to the height of the jth cut rectangle. The cut-rectangles are reinserted at their origin positions.

This step is done for all sets B^w_i. The rounding method for the long rectangles in L^w is almost the same. However, we do not have the property that the rectangles in L^w have all the same width. Therefore, we have to slice the long rectangles vertically. We also sort the rectangles of set L^w according to non-decreasing heights. $w(L^w)$ denotes the total width of the rectangles in set L^w. We divide the set L^w into subsets $L^w_1, \ldots, L^w_{1/\delta^2}$ of the same total width, by splitting rectangles vertically if necessary. The subset L^w_1 contains the largest and the subset L^w_{1/δ^2} the shortest rectangles. The rectangles in each subset have a total width $w(L^w_i)$ of $\delta^2 \cdot w(L^w)$. We round up the heights of the rectangles in each subset to the height of the largest rectangle in it. The rectangles in subset L^w_1 are packed later into additional bins. The rectangles of the remaining subsets are packed on the positions where the rectangles of the previous subset have been. Again, we split the rectangles vertically if necessary.

It is left to pack the rectangles in the subsets $L^w_1, B^w_{2/\delta,1}, \ldots, B^w_{2/\delta^2,1}$ into additional bins. The total width of all rectangles in all sets $w(L^w \cup B^w_{2/\delta} \cup \ldots \cup B^w_{2/\delta^2})$ is at most $1/\delta \cdot p_1$, since each rectangle has a height of at least δ and they would not fit into p_1 bins otherwise. Hence,

$$w(L^w_1) + \sum_{i=2/\delta}^{2\delta^2} w(B^w_{i,1}) \leq \delta^2 \cdot w(L^w) + \sum_{i=2/\delta}^{2\delta^2} \delta^2 \cdot w(B^w_i) =$$

$$\delta^2 \cdot (w(L^w) + \sum_{i=2/\delta}^{2/\delta^2} w(B^w_i)) \leq \delta^2 \cdot 1/\delta \cdot p_1 = \delta \cdot p_1.$$

We pack these rectangles greedily on the floor of additional bins. We start by packing each big rectangle of width larger than $1/2$ into one additional bin. This bin is closed and we do not pack further rectangles into it. The remaining big rectangles have a width of at most $1/2$. We pack these rectangles greedily on the floor of the bin, until the next big rectangle does not fit. Then a new bin is opened and we continue with the packing. Afterwards, we pack the long rectangles with the same method. We secure, that the rectangles in each but the last bin have a total width of at least $1/2$. In total, we need at most $\lceil 2(\delta \cdot p_1) \rceil \leq 2\delta \cdot p_1 + 1$ bins while each packing satisfies Property 1, since each big rectangle is placed on a multiple of $\delta^2/2$.

The same steps are done analogously for rounding the widths of the big and wide rectangles that are packed in the bins $B_{p_1+1}, \ldots, B_k$. Therefore, we need $2\delta \cdot (1 - p_1) + 1$ additional bins. In doing so, we use similar subsets $B^h_{i,j}$ and W^h_j for $i \in \{2/\delta, \ldots, 2/\delta^2\}$ and $j \in \{1, \ldots, 1/\delta^2\}$.

With the above discussion we obtain the following result.

Lemma 23. *We round up the heights of the long and big rectangles in each set L^w and B^w_i, and the widths of the wide and big rectangles in each set W^h and B^h_i for $i \in \{2/\delta, \ldots, 2/\delta^2\}$ to at most $1/\delta^2$ values. Therefore, we need $2\delta \cdot k + 2$ additional bins.*

Rounding Big and Long Rectangles with Rotations In the version that allows rotation, we only have bins with a packing that satisfies Property 1. In this version, we employ Theorem' 4 and obtain a solution in $k' \leq (3/2 + \delta) \cdot \mathrm{OPT} + 21$ bins. Consequently, we have the following lemma.

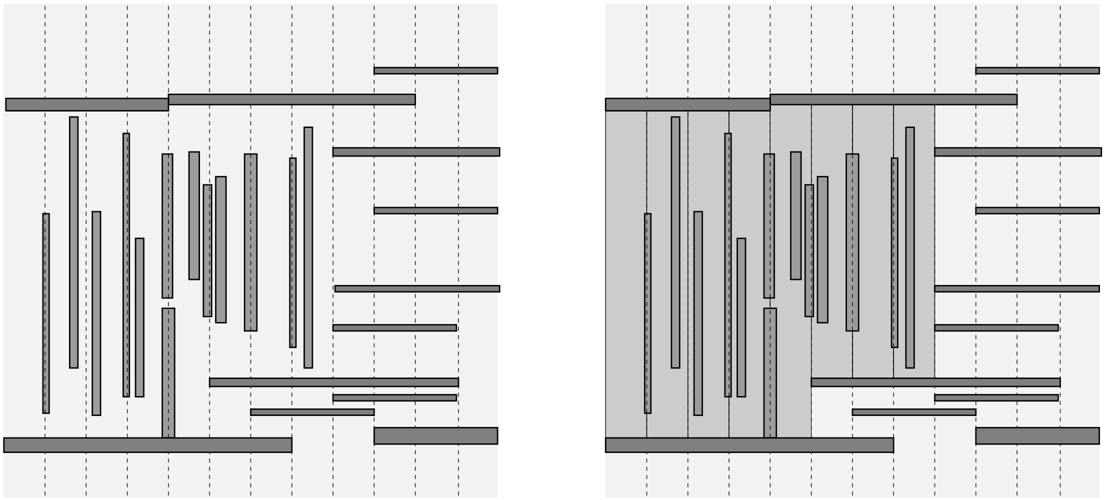
Lemma' 23. *We round up the heights of the long and big rectangles in each set L^w and B^w_i for $i \in \{2/\delta, \ldots, 2/\delta^2\}$ to at most $1/\delta^2$ values and need therefore $2\delta \cdot k' + 1$ additional bins.*

2.3.2 Containers for the Wide and Long Rectangles

In this section, we construct rectangular containers for the wide and long rectangles. To do this, we employ some techniques of Jansen & Solis-Oba [16]. We explain these steps only for the bins $B_1, \ldots, B_{p_1}$ of Type 1 as it works analogously for the remaining bins. We define a set of long containers C^w_L for the long rectangles and a set of wide containers C^w_W for the wide rectangles. We focus on one bin B_i, for $i \in \{1, \ldots, p_1\}$.

We define slots of width $\delta^2/2$ by drawing vertical lines on each multiple of $\delta^2/2$ in the bin B_i for the long containers. A small or long rectangle that intersects one of these vertical lines is vertically cut by it. Each wide and big rectangle intersects either a slot completely or it does not intersect it at all. Hence, a long container is a part of a slot which is bounded at the top and at the bottom by a wide or a big rectangle or the top or the floor of the bin. We only consider containers with at least one long rectangle, i.e. the height of each long container is at least δ. The width of each long container is $\delta^2/2$ (cf. Figure 26). As an upper bound there are at most $1/\delta - 1$ long containers in each slot separated by wide rectangles of small height. Hence, in bin B_i there are at most $(1/\delta - 1) \cdot 2/\delta^2 = 2/\delta^3 - 2/\delta^2$ long containers. This is an upper bound when there are no big rectangles situated in this bin. Each big rectangle that is situated in this bin decreases the number of long containers by $2/\delta$ since it intersects $2/\delta$ slots and it has a height of at least δ.

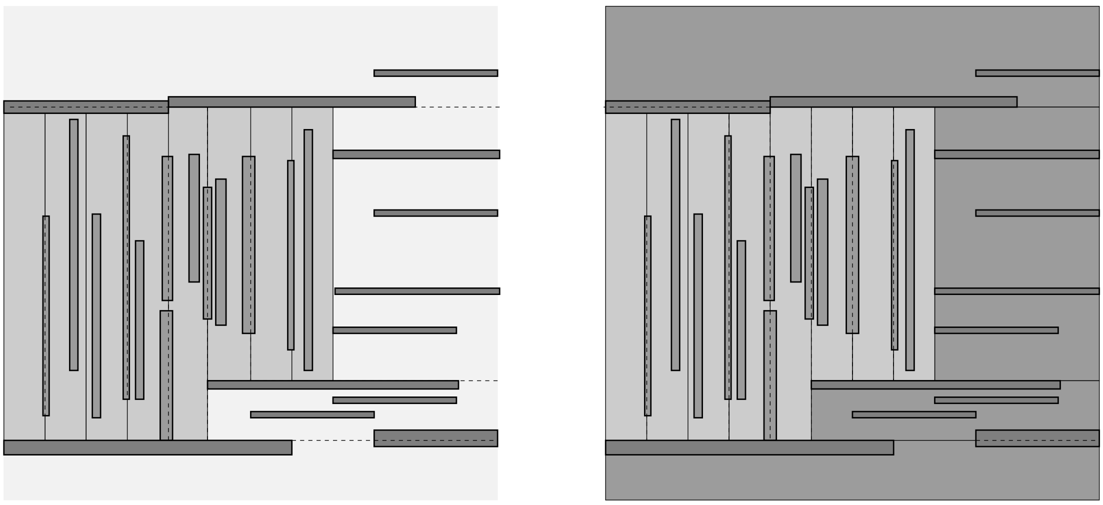
Afterwards, we construct horizontal lines by extending the upper and lower edge of each big rectangle and each long container in both directions until it hits another big rectangle, a long container or the sides of the bin. Wide and small rectangles are horizontally cut by these lines. The horizontal lines are the upper and lower edges of the wide containers. Since they are bounded at the sides by big rectangles, long containers or the sides of the bin, the wide containers always have a width of a multiple of $\delta^2/2$. There are at most $2/\delta^3 - 2/\delta^2$ big rectangles and long containers in the solution, hence we extend at most $2/\delta^3 - 2/\delta^2$ upper and lower edges. Furthermore, we have two additional lines with the bottom and the top of the bin. Each extended upper edge is a possible lower edge of one wide container. Each extended lower edge may be the lower edges of two wide containers, one on the left and one on the right side. Furthermore, the bottom of the bin is a possible lower edge of one wide container. It follows



(a) Part of a packing in one bin (b) Construction of long containers

Figure 26: Construction of long containers; long and small rectangles are sliced vertically

that there are at most $3 \cdot (2/\delta^3 - 2/\delta^2) + 1 = 6/\delta^3 - 6/\delta^2 + 1$wide containers in bin B_i. At this moment, the complete region of the bin is filled with big rectangles, long containers and wide containers. There is no empty space left, hence all small rectangles are fractionally in the long and wide containers (cf. Figure 27). This construction is done for all bins $B_1, \ldots, B_{p_1}$.



(a) Drawing horizontal lines (b) Construction of wide containers

Figure 27: Construction of wide containers; wide and small rectangles are sliced horizontally

2.3.3 Rounding the Wide and Long Containers

Let c_w be one of the wide containers in C_W^w. The height of c_w is now reduced so that it has a height of a multiple of δ^4. We cut the uppermost wide and small rectangles horizontally by the new height. The (fractions of the) wide and small rectangles of height less than δ^4 that do not fit in the reduced container are placed next to each other into an additional bin. We do this with all wide containers in the bins $B_1, \dots, B_{p_1}$. Since all rectangles have a height of less than δ^4 we are able to place the rectangles of $1/\delta^4$ containers on top of each other into one bin. In total, we have less than $6/\delta^3 \cdot p_1$ wide containers and hence we need $\lceil \delta^4 \cdot 6/\delta^3 \cdot p_1 \rceil \leq 6\delta \cdot p_1 + 1$ additional bins. These bins contain only wide and small rectangles, whereas the wide rectangles are placed on x-coordinates of a multiple of $\delta^2/2$. Hence, the packing of these bins satisfy Property 1. We treat each additional bin as one wide container of height and width 1.

After the construction of the wide containers we round down the heights of the long containers. We focus one long container c_ℓ in C_L^w in the bins $B_1, \dots, B_{p_1}$. We remove all short rectangles in this container and shift all long rectangles vertically down, so that they all touch either the bottom of the container or another long rectangle. The total used height in this container is a combination of the heights of the long rectangles. Since the heights of the long rectangles are rounded, the possible number of heights is bounded by a polynomial in the length of the input. If the remaining space on top of the uppermost long rectangle in the container c_ℓ has a height of at least δ^4, then we are able to round the height of the container down to the next multiple of δ^4. If the remaining space is less than δ^4 we round the height of c_ℓ down to the height of the top edge of the uppermost long rectangle in c_ℓ. Hence, the height of c_ℓ is either a combination of the rounded heights of the long rectangles or a multiple of δ^4.

It is not possible to reinsert all small rectangles. Hence, we pack them fractionally into the reduced container until the free space is exceeded. The remaining rectangles are packed fractionally into additional bins. The total area loss for each long container is at most $\delta^4 \cdot \delta^2/2 = \delta^6/2$. There are at most $2/\delta^3$ long containers in each bin, hence the total area is at most $\delta^6/2 \cdot 2/\delta^3 \cdot p_1 = \delta^3 \cdot p_1$. Thus, we need at most $\lceil \delta^3 \cdot p_1 \rceil \leq \delta^3 \cdot p_1 + 1$ additional bins. These additional bins contain only small rectangles and the packing satisfies Property 1. We treat each bin as one wide container of height and width 1, in order to ensure that all small rectangles are packed into containers. By this construction, the total number of wide containers is at most $(6/\delta^3 - 6/\delta^2 + 1) \cdot p_1 + 6\delta \cdot p_1 + 1 + \delta^3 \cdot p_1 + 1 = (6/\delta^3 - 6/\delta^2 + 6\delta + \delta^3 + 1) \cdot p_1 + 2 < 6/\delta^3 \cdot p_1 + 2$.

We also construct long containers in the $2\delta \cdot p_1 + 1$ additional bins that are needed to round the heights of the long rectangles (cf. Lemma 23). The long rectangles are placed on the bottom of these bins and there are no small rectangles in it. Furthermore, we packed them after we packed the big rectangles so that there is at most one bin that contains big and long rectangles. The other bins contain either only big rectangles or only long rectangles. We draw also vertical lines in the bins that contain long rectangles at each multiple of $\delta^2/2$ and cut the long rectangles intersecting these lines. These lines form already the long containers in these bins, since there are no rectangles on top of the long rectangles. Therefore, we have in these bins at most $(2\delta \cdot p_1 + 1) \cdot 2/\delta^2 = 4/\delta \cdot p_1 + 2/\delta^2$ long containers of width $\delta^2/2$ and height 1.

There is still a huge number of different types of the long containers, since we have $1/\delta^2$ different heights of the long rectangles and hence at least $(1/\delta^2)^{1/\delta}$ possibilities for the heights. In order to reduce this number to $1/\delta^2$ different heights, we use the same rounding technique as for the big rectangles.

Let $k_\ell \leq (2/\delta^3 - 2/\delta^2) \cdot p_1 + 4/\delta \cdot p_1 + 2/\delta^2 \leq 2/\delta^3 \cdot p_1 + 2/\delta^2$ be the total number of long containers in C_L^w, i.e. in the bins of Type 1. We sort all k_ℓ containers according to non-decreasing heights and denote the sorted containers by $c_1, \dots, c_{k_\ell}$. We partition the long containers into at most $1/\delta^2$ subsets of $\lfloor \delta^2 k_\ell \rfloor$ containers. The construction is analogous to the rounding of the big rectangles by calling c_1 the first cut-container and assigning the next $\lfloor \delta^2 k_\ell \rfloor$ containers to the first subset and so on. The heights in each subset are rounded up to the height of the previous cut-container. The containers in the first subset are moved into additional bins, whereas we are able to pack $2/\delta^2$ containers next to each other at the bottom in one bin. Hence, we need at most $\lceil \lfloor \delta^2 k_\ell \rfloor \cdot \delta^2/2 \rceil \leq \lceil \delta^2 \cdot (2/\delta^3 \cdot p_1 + 2/\delta^2) \cdot \delta^2/2 \rceil \leq \delta \cdot p_1 + \delta^2 + 1$ additional bins. The long containers in the remaining subsets are packed on the position of a long container in the previous subset and the cut-container are placed at their origin position.

We do the analogous steps for the bins $B_{p_1+1}, \dots, B_k$ and achieve the set of wide containers C_W^h and the set of long containers C_L^h with analogous bounds by replacing p_1 with $k - p_1$. This leads us to the following result:

Lemma 24. *Suppose we have a packing without medium rectangles in k bins, while the packing of each of them satisfies either Property 1 or Property 2. If the long and wide rectangles are rounded according to Lemma 23, then we are able to pack the wide, long and small rectangles fractionally into containers with at most $8\delta \cdot k + 2\delta^2 + 6$ additional bins. The rectangles of W^w and W^h are sliced horizontally and packed into wide containers of C_W^w and C_W^h, respectively. The long rectangles of L^w and L^h are sliced vertically and packed into long containers of C_L^w and C_L^h. All small rectangles are packed fractionally (vertically and horizontally sliced) into these containers. The containers have the following properties:*

3.1. *there are at most $6/\delta^3 \cdot p_1 + 2$ wide containers in C_W^w, that have a width of a multiple of $\delta^2/2$ and a height of a multiple of δ^4.*
3.2. *there at most $2/\delta^3 \cdot (k - p_1) + 2/\delta^2$ wide containers in C_W^h, of at most $1/\delta^2$ different widths of either a multiple of δ^4 or a combination of the rounded widths of the wide rectangles in W^h and of a height $\delta^2/2$*
3.3. *there are at most $6/\delta^3 \cdot (k - p_1) + 2$ long containers in C_L^h, that have a height of a multiple of $\delta^2/2$ and a width of a multiple of δ^4.*
3.4. *there are at most $2/\delta^3 \cdot p_1 + 2/\delta^2$ long containers in C_L^w, of at most $1/\delta^2$ different heights of either a multiple of δ^4 or a combination of the rounded heights of the long rectangles in L^w and of a width $\delta^2/2$*

Proof. By rounding the heights of the wide containers in C_W^w in the bins of Type 1 and the long containers in C_L^h in the bins of Type 2 to a multiple of δ^4 we need $6\delta \cdot p_1 + 1 + 6\delta \cdot (k - p_1) + 1 = 6\delta \cdot k + 2$ additional bins. The heights of the long containers in C_L^w in the bins of Type 1 are rounded to at most $1/\delta^2$ values of either a multiple of δ^4 or a combination of the rounded heights of the long rectangles. Therefore, we need $\delta^3 \cdot p_1 + 1 + \delta \cdot p_1 + \delta^2 + 1 = (\delta + \delta^3) \cdot p_1 + \delta^2 + 2$ additional bins. The analogous steps for rounding the widths of the wide containers in C_W^h in the bins of Type 2 need $(\delta + \delta^3) \cdot (k - p_1) + \delta^2 + 2$ additional bins and hence together $(\delta + \delta^3) \cdot k + 2\delta^2 + 4$ additional bins. The total number of additional bins is thus $(6\delta + \delta + \delta^3) \cdot k + 2\delta^2 + 6 \leq 8\delta \cdot k + 2\delta^2 + 6$. □

To conclude, we obtain the following result:

Theorem 6. *Given an optimal solution of an instance I into OPT bins. We are able to round up the widths and heights of the rectangles and to modify the solution so that it fits into at most $(3/2 + 25\varepsilon') \cdot \mathrm{OPT} + 55$ bins, while all medium rectangles are packed into $3\varepsilon' \cdot \mathrm{OPT} + 2$ bins and $3/2\mathrm{OPT} + 22\delta\mathrm{OPT} + 53$ bins have a packing that satisfies either Property 1 or Property 2. Furthermore, the heights of the long and big rectangles in each set L^w and B_i^w, and the widths of the wide and big rectangles in each set W^h and B_i^h for $i \in \{2/\delta, \dots, 2/\delta^2\}$ are rounded up to at most $1/\delta^2$ values. The wide and long rectangles are sliced horizontally and vertically, respectively. They are packed into wide and long containers with the Properties 3.1-3.4. The small rectangles are packed fractionally into the wide and long containers.*

Proof. In the first step we employ Theorem 4 and need in total $k \leq (3/2 + 5\delta)\mathrm{OPT} + 37$ bins. Afterwards, we pack all medium rectangles into $3\varepsilon' \cdot \mathrm{OPT} + 2$ additional bins with Lemma 22. We round up the big, long and wide rectangles with Lemma 23 and need $2\delta \cdot k + 2$ additional bins. The wide and long rectangles are packed into long and wide containers with Lemma 24. Therefore, we need $8\delta \cdot k + 2\delta^2 + 6$ additional bins. It holds $\delta \leq 1/48$, hence in total we need at most

$$\begin{aligned}(1 + 10\delta) \cdot ((3/2 + 5\delta) \cdot \mathrm{OPT} + 37) + 2\delta^2 + 8 &= \\ 3/2 \cdot \mathrm{OPT} + 5\delta \cdot \mathrm{OPT} + 15\delta \cdot \mathrm{OPT} + 50\delta^2 \cdot \mathrm{OPT} + 37 + 370\delta + 2\delta^2 + 8 &\leq \\ 3/2 \cdot \mathrm{OPT} + 20\delta \cdot \mathrm{OPT} + 50\delta/48 \cdot \mathrm{OPT} + 45 + 370/48 + 2/48^2 &\leq \\ 3/2 \cdot \mathrm{OPT} + 22\delta \cdot \mathrm{OPT} + 53 &\end{aligned}$$

bins that have a packing that satisfies either Property 1 or Property 2. Since $\delta \leq \varepsilon'$, we have at most $(3/2 + 25\varepsilon') \cdot \mathrm{OPT} + 55$ bins (including the bins for the medium rectangles). □

This finishes our analysis and modification of an optimal solution. It enables us to construct an algorithm that computes a packing that almost matches the modified solution.

Rounding the Wide and Long Containers with Rotations The steps are analogous for the versions with rotations.

Lemma' 24. *Suppose we have a packing without medium rectangles in k' bins, while the packing of each of them satisfies Property 1. If the long and wide rectangles are rounded according to Lemma 23, then we are able to pack the wide, long and small rectangles fractionally into containers with at most $8\delta \cdot k' + \delta^2 + 3$ additional bins. The rectangles of W^w are sliced horizontally and packed into wide containers of C^w_W. The long rectangles of L^w are sliced vertically and packed into long containers of C^w_L. All small rectangles are packed fractionally (vertically and horizontally sliced) into these containers. The containers have the following properties:*

3.5. *there are at most $6/\delta^3 \cdot k' + 2$ wide containers in C^w_W, that have a width of a multiple of $\delta^2/2$ and a height of a multiple of δ^4.*
3.6. *there are at most $2/\delta^3 \cdot k' + 2/\delta^2$ long containers in C^w_L, of at most $1/\delta^2$ different heights of either a multiple of δ^4 or a combination of the rounded heights of the long rectangles in L^w and of a width of $\delta^2/2$*

To conclude, we obtain a better additional constant for modifying the packing.

Theorem' 6. *Given an optimal solution of an instance I into OPT bins. We are able to rotate and round up the widths and heights of the rectangles and to modify the solution so that it fits into at most $(3/2 + 25\varepsilon') \cdot \mathrm{OPT} + 31$ bins, while all medium rectangles are packed into $3\varepsilon' \cdot \mathrm{OPT} + 1$ bins and $3/2 \cdot \mathrm{OPT} + 22\delta \cdot \mathrm{OPT} + 30$ bins have a packing that satisfies Property 1. Furthermore, the heights of the long and big rectangles in the sets L^w and B^w_i for $i \in \{2/\delta, \ldots, 2/\delta^2\}$ are rounded to at most $1/\delta^2$ values. The wide and long rectangles are sliced horizontally and vertically, respectively. They are packed into wide and long containers with the Property 3.5 and Property 3.6. The small rectangles are packed fractionally into the wide and long containers.*

Proof. We adopt Theorem' 4 with an optimal solution and obtain $k' \leq (3/2 + 5\delta)\mathrm{OPT} + 21$ bins. We round up the big and long rectangles according to Lemma' 23 and need $2\delta \cdot k' + 1$ additional bins. For rounding the containers with Lemma' 24, we need $8\delta \cdot k' + \delta^2 + 3$ additional bins. The additional constant is therefore, $21 + 10\delta \cdot 21 + \delta^2 + 1 + 3 \leq 25 + 210/48 + 1/48^2 \leq 25 + 5 = 30$. Together with the $3\varepsilon' \cdot \mathrm{OPT} + 1$ additional bins that are used to pack the medium rectangles in Lemma' 22, we obtain $(3/2 + 25\varepsilon') \cdot \mathrm{OPT} + 31$ bins. □

3 Algorithm

In the last sections we modified an optimal solution in order to achieve a simpler structure. In this section we describe our algorithm. The algorithm works in two parts. The first part is to transform an instance I of n rectangles into the rounded instance, the second part is to pack the rounded rectangles into the bins.

3.1 Transform an Instance I

For dual approximation we use binary search to find the optimum $\mathrm{OPT} = \mathrm{OPT}(I)$ of I. In each iteration with a candidate OPT' for OPT we either find a solution with at most $(3/2 + 22\delta) \cdot \mathrm{OPT}' + 69$ bins or conclude that $\mathrm{OPT}' < \mathrm{OPT}$. In the first case we decrease OPT' in order to try if there is a solution with less bins and in the second case we increase OPT'. The upper bound for OPT is the number of rectangles in the instance I, the lower bound is the total area of the rectangles. In the following, we assume that we found an $\mathrm{OPT}' \leq \mathrm{OPT}$ so that our algorithm is able to compute a solution. In the next step we are able to set δ according to Lemma 21 and divide the instance into big, long, wide, small and medium rectangles. We pack all medium rectangles with Steinberg's Theorem 5 into $3\varepsilon'\mathrm{OPT} + 2$ additional bins (cf. Lemma 22). Afterwards, we have to distinguish whether the width or the height of each big rectangle is rounded up to a multiple of $\delta^2/2$. In other words, we have to distinguish, whether a rectangle belongs to a bin of Type 1, i.e. it is in a set B_i^w, or to a bin of Type 2, i.e. it is in a set B_i^h, for one $i \in \{2/\delta, \dots, 2/\delta^2\}$.

In the version that allows rotation, we only have bins of Type 1. However, we do not know which side we have to round up to the next largest multiple of $\delta^2/2$. Therefore, we have to solve a similar problem.

3.1.1 Transform Big Rectangles

Let $i \in \{2/\delta, \dots, 2/\delta^2\}$ and $j \in \{1, \dots, 1/\delta^2\}$. We guess the number of big rectangles that are rounded to each width of $i\delta^2/2$ and to each height of $i\delta^2/2$. In other words, we guess the cardinality of the sets B_i^w and B_i^h. This can be done by choosing less than $2 \cdot 2/\delta^2 = 4/\delta^2$ values out of n. With the guessed cardinality we compute the number of the at most $1/\delta^2$ subsets $B_{i,j}^w$ and $B_{i,j}^h$ and also the number of the at most $1/\delta^2$ cut-rectangles. We guess the cut-rectangles by choosing $2 \cdot 2/\delta^2 \cdot 1/\delta^2 = 4/\delta^4$ rectangles out of n possible rectangles. We denote the number of rectangles that are rounded up to the width $i \cdot \delta^2/2$ and to the height of the jth cut-rectangle in B_i^w by $n_{i,j}^w$ and we denote the number of rectangles that are rounded up to the height $i \cdot \delta^2/2$ and to the width of the jth cut-rectangle in B_i^h by $n_{i,j}^h$.

These values give us the structure of the subsets of the Section 2.3.1, since we know the rounded heights and widths of the big rectangles and the number of rectangles with these side lengths. To assign big rectangles to these subsets we set up a flow network $G = (V, E)$ with the set V of vertices and the set E of edges. Each big rectangle of I has a corresponding node in this network and is connected with an edge of capacity 1 to the source s. Either the width or the height of a big rectangle is rounded up to the next multiple of $\delta^2/2$. If this is decided, we know also the corresponding subset where the rectangle belongs to since it has to be between the heights or the widths of two cut-rectangles. However, when the height or the width is exactly that one of a cut-rectangle it could belong to possibly more subsets. For each subset $B_{i,j}^w$ and $B_{i,j}^h$ we have one node in the network and connect it to each rectangle that may belong to it with an edge of capacity 1. There are at most $2 \cdot 2/\delta^2 \cdot 1/\delta^2 = 4/\delta^4$ subset-nodes and each big-

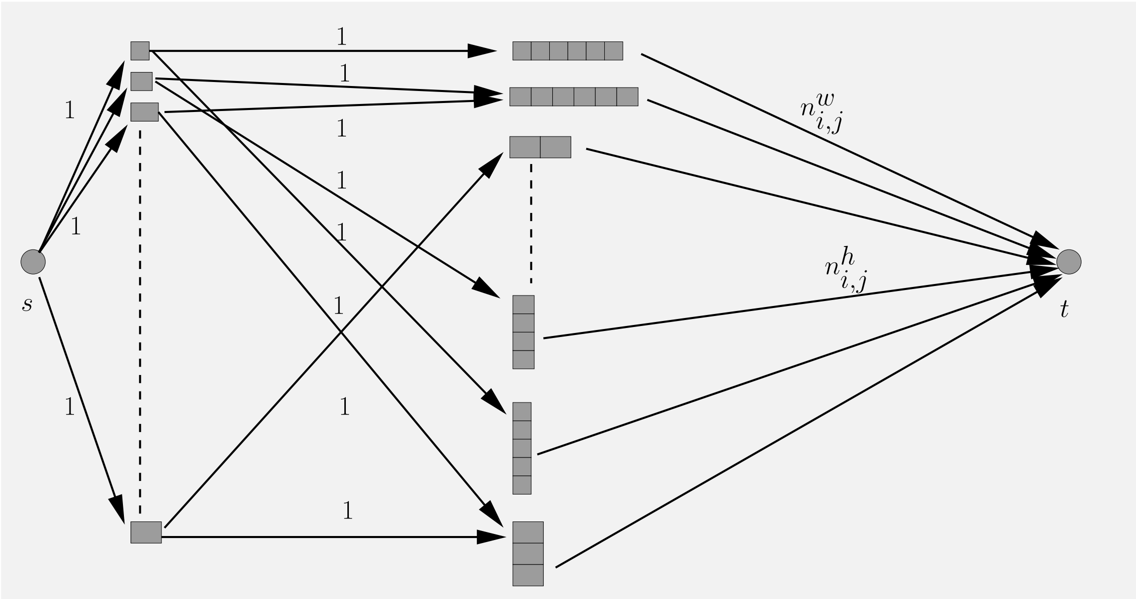


Figure 28: The flow-network

rectangle-node is connected to at most $2 \cdot 1/\delta^2 = 2/\delta^2$ of them, when all cut-rectangles have the same height and width, respectively. Each subset-node $B^w_{i,j}$ and $B^h_{i,j}$ is connected with an edge to the sink t of capacity $n^w_{i,j}$ and $n^h_{i,j}$, respectively (cf. Figure 28). The total number of vertices is in total $|V| \leq 2+n+4/\delta^4$. The number of edges is $n+n\cdot 2/\delta^2+4/\delta^4$. We find a flow with the algorithm of Dinic [8] in time $\mathcal{O}(|E|\cdot|V|^2) = \mathcal{O}((n+(2n)/\delta^2+4/\delta^4)\cdot(2+n+4/\delta^4)^2) = \mathcal{O}(n^3/\delta^2+n^2/\delta^6+n/\delta^{10}+1/\delta^{14})$. If there is a possible assignment of big rectangles to the subsets then there is a flow with the same value as the number of big rectangles, hence each edge at the source s is satisfied. If there is no flow with this value at all, there exists no assignment of big rectangles to the subsets and we have to try another guess.

Transform Big Rectangles with Rotations In the version that allows rotation, we only have subsets $B^w_{i,j}$. In this setting we connect each big rectangle with the corresponding subset-nodes before and after rotating it by $90°$. When we find a flow that satisfies each edge from the source we have assigned all big rectangles to the corresponding subsets and have decided whether we have to rotate a big rectangle, or not.

3.1.2 Transform Wide and Long Rectangles

We explain these steps for the wide rectangles, the transformation for the long rectangles is analogous. We have to decide whether a wide rectangle belongs to a bin of Type 1, i.e. to one set $W^w_{2/\delta},\dots,W^w_{2/\delta^2}$, or to a bin of Type 2, i.e. to one set $W^h_1,\dots,W^h_{1/\delta^2}$. To do this, we guess the $1/\delta^2$ widths $w_{c_1},\dots,w_{c_{1/\delta^2}}$ that are used to round up the rectangles in the sets $W^h_1,\dots,W^h_{1/\delta^2}$. This is done by choosing $1/\delta^2$ rectangles $r_{c_1},\dots,r_{c_{1/\delta^2}}$ out of n rectangles. Remember that $w_{c_1} \geq w_{c_2} \geq \dots \geq w_{c_{1/\delta^2}}$. The total heights of the sets are identical, therefore we guess the total height of the whole set W^h approximately and divide it by $1/\delta^2$ to obtain the total height of each subset. The total height of all wide rectangles is bounded by $\delta^4 \cdot n$ since each wide rectangle has a height of at most δ^4. We choose 1 integral value i_0 out of $1/\delta^4 \cdot \delta^4 \cdot n = n$, so that $i_0 \cdot \delta^4 \leq h(W^h) < (i_0+1)\cdot\delta^4$ holds. This leads to an approximately guessed structure of the sets $W_1^h, \dots, W_{1/\delta^2}^h$, since we know the widths and the heights.

The widths of the wide rectangles that are in the sets $W_{2/\delta}^w, \dots, W_{2/\delta^2}^w$ are rounded up to the next multiple of $\delta^2/2$, hence the width of each rectangle in the set W_j^w is rounded up to $j\delta^2/2$. We guess approximately the total height of the rectangles in each set. Therefore, we choose $2/\delta^2 - 2/\delta + 1$ integral values $i_{2/\delta}, \dots, i_{2/\delta^2}$ out of $1/\delta^4 \cdot \delta^4 \cdot n = n$, so that $i_j \cdot \delta^4 \leq h(W_j^w) < (i_j+1)\delta^4$ holds for all $j \in \{2/\delta, \dots, 2/\delta^2\}$. Consequently, we have the structure of all sets of wide rectangles and we have to assign the wide rectangles into them.

Note that $(i_0+1)\delta^4 + \sum_{z_1=2/\delta}^{2/\delta^2}(i_{z_1}+1)\delta^4$ is larger than the total height of all wide rectangles in the instance.

Assigning the Rectangles We sort the wide rectangles according to non-increasing widths. Let r_w be a wide rectangle. r_w is a candidate for the set W_j^w, if r_w has a width of $w_w \in ((j-1)\delta^2/2, j\delta^2/2]$, for $j \in \{2/\delta+1, \dots, 2/\delta^2\}$ and a candidate for $W_{2/\delta}^w$ if $w_w = \delta$. Furthermore, this rectangle is a candidate for W_j^h, if $w_w \in [w_{c_j}, w_{c_{j+1}}]$, for $j \in \{1, \dots, 1/\delta^2 - 1\}$ or for W_{1/δ^2}^h if $\delta \leq w_w \leq w_{c_{1/\delta^2}}$.

First, we assign the wide rectangles greedily to the set W_{2/δ^2}^w. Therefore, we take the widest candidates for this set until the total height exceeds $(i_{2/\delta^2}+1)\cdot\delta^4$, i.e. the total height of these rectangles is at most $(i_{2/\delta^2}+1)\cdot\delta^4 + \delta^4$. If we run out of candidates before we reach this height, we take the widest candidates for W_{2/δ^2-1}^w and so on. This is repeated for all sets $W_{2/\delta^2-1}^w, \dots, W_{2/\delta}^w$. We selected the widest candidates for these sets (cf. Figure 29). The remaining wide rectangles are greedily assigned in the same way into the sets $W_1^h, \dots, W_{1/\delta^2}^h$.

Lemma 25. *For the right guess of the values $i_0, i_{2/\delta}, \dots, i_{2/\delta^2}$ and for the right guess of the rectangles $r_{c_1}, \dots, r_{c_{1/\delta^2}}$ we assign each wide rectangle to one of the sets $W_{2/\delta}^w, \dots, W_{2/\delta^2}^w$ and $W_1^h, \dots, W_{1/\delta^2}^h$.*

Proof. Suppose by contradiction that there are rectangles that can not be assigned to one set for the right guess of the values $i_0, i_{2/\delta}, \dots, i_{2/\delta^2}$ and for the right guess of the rectangles $r_{c_1}, \dots, r_{c_{1/\delta^2}}$. Let the rectangle r_w be the widest rectangle among them. Suppose that r_w is a candidate for W_i^w and a candidate for W_j^h, for the largest possible j if r_w is a candidate for several sets. All rectangles that have a width of at least w_w, including r_w, have to fit fractionally into the sets $W_{2/\delta^2}^w, \dots, W_i^w$ and the sets $W_1^h, \dots, W_j^h$. Let X denote the set of wide rectangles that have a width of at least w_w, including r_w. Consequently, we have

$$h(X) \leq \sum_{z_1=i}^{2/\delta^2} h(W_{z_1}^w) + \sum_{z_2=1}^{j} h(W_{z_2}^h) \leq \sum_{z_1=i}^{2/\delta^2}(i_{z_1}+1)\delta^4 + \sum_{z_2=1}^{j}(i_0+1)\delta^4\cdot\delta^2$$

On the other hand, the sets have to be completely full, since we are not able to pack r_w into one of these sets, i.e. the total height of the rectangles of $W_{z_1}^w$ is larger than $(i_{z_1}+1)\delta^4$ and the total height of the rectangles in $W_{z_2}^h$ is larger than $(i_0+1)\delta^4\cdot\delta^2$, for all $z_1 \in \{i, \dots, 2/\delta^2\}$ and $z_2 \in \{1, \dots, j\}$. It follows that $h(X) > \sum_{z_1=i}^{2/\delta^2}(i_{z_1}+1)\delta^4 + \sum_{z_2=1}^{j}(i_0+1)\delta^4\cdot\delta^2$ which is a contradiction. □

Consequently, all rectangles have to fit into these sets. Afterwards, we remove the shortest rectangles in each set $W_{2/\delta}^w, \dots, W_{2/\delta^2}^w$ and $W_1^h, \dots, W_{1/\delta^2}^h$, in order to secure that the total height is at most $i_j\delta^4$ and $i_0\delta^4\cdot\delta^2$, respectively. Therefore, we have to remove wide rectangles of a total height of at most $3\delta^4$ for each set since we have to reduce the total height by at most

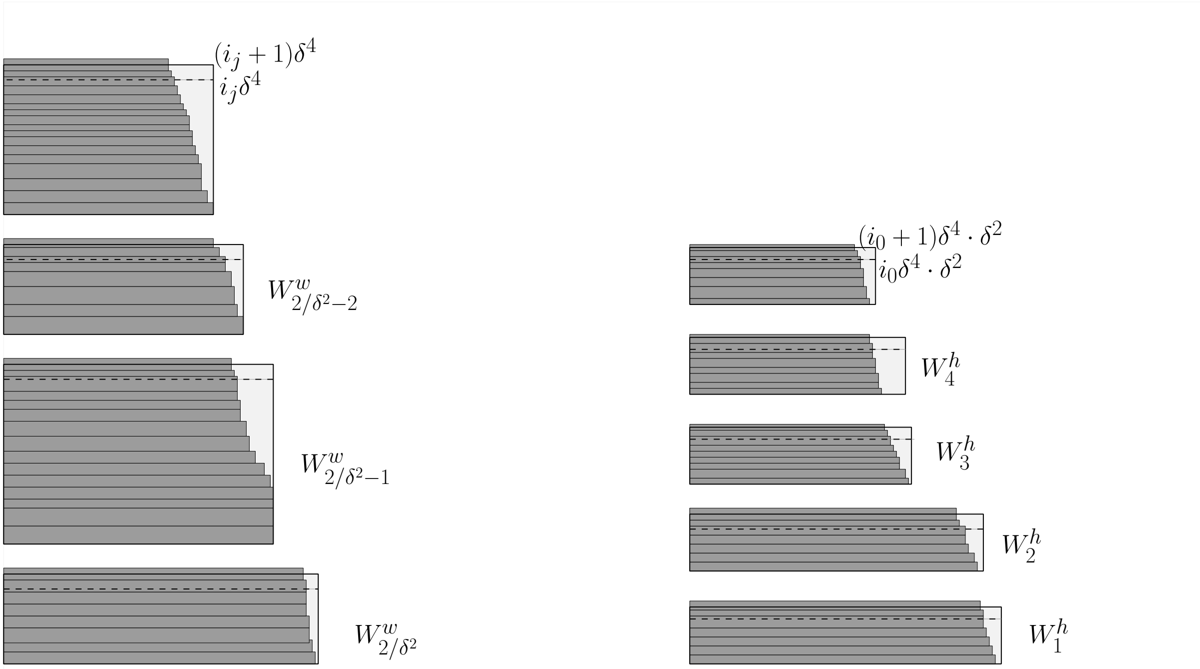


Figure 29: A greedy assignment of wide rectangles; sort the rectangles by their widths, pack them into the sets $W^w_{2/\delta^2}, \ldots, W^w_{2/\delta}$ until the last rectangle exceeds $(i_j+1)\delta^4$; afterwards pack the remaining rectangles into the sets $W^h_1, \ldots, W^h_{1/\delta^2}$.

$2\delta^4$ and we have to remove the wide rectangle of height δ^4 that is cut by the new height. In total, we have wide rectangles of a total height of $(2/\delta^2 - 2\delta + 1 + 1/\delta^2) \cdot 3\delta^4 \leq 9\delta^2$. These rectangles fit into one additional bin by packing them on top of each other. The same steps are done to assign the long rectangles to the sets $L^w_1, \ldots, L^w_{1/\delta^2}$ and $L^h_{2/\delta}, \ldots, L^h_{2/\delta^2}$ and we need one additional bin.

To conclude, we guess $2 \cdot (1/\delta^2 + 1 + 2/\delta^2 - 2/\delta + 1) \leq 6/\delta^2$ rectangles and values out of n values and need two additional bins to transform the wide and long rectangles.

Transform Long and Wide Rectangles with Rotations In the version that allows rotation, we rotate each long rectangle in order to have only wide rectangles. We assign them to the sets $W^w_{2/\delta}, \ldots, W^w_{2/\delta^2}$ and $L^w_1, \ldots, L^w_{1/\delta^2}$. As above, we guess approximately the total width of L^w and the total height of the sets $W^w_{2/\delta}, \ldots, W^w_{2/\delta^2}$. Furthermore, we guess the heights of the cut-rectangles, in order to get the structure of the sets $L^w_1, \ldots, L^w_{1/\delta^2}$. Therefore, we choose $1/\delta^2$ rectangles and take the widths of the selected rectangles as the heights of the cut-rectangles, i.e. we rotate the selected rectangles. Afterwards, we greedily assign the wide rectangles to the sets $W^w_{2/\delta^2}, \ldots, W^w_{2/\delta}$. The remaining rectangles are rotated and they are assigned to the sets $L^w_1, \ldots, L^w_{1/\delta^2}$. The rectangles that have to be removed fit into one additional bin. These are the only algorithmic differences between the version with and without rotations and the remaining steps work for both versions. However, in the following we continue to state the minor differences to improve the additive constant.

3.1.3 Construct the Containers

We do the following steps for the long and wide containers that are packed in the bins of Type 1, the construction of the containers in the bins of Type 2 is analogous. Each wide container of C^w_W has a height of a multiple of δ^4 and a width of a multiple of $\delta^2/2$ (cf. Lemma 24). Hence, there are at most $1/\delta^4 \cdot 2/\delta^2 = 2/\delta^6$ different types of wide containers in the solution. We guess the number $n^w_{i,j}$ of the wide containers of each width $i\delta^2/2$ and each height $j\delta^4$ by choosing $2/\delta^6$ values out of n (we suppose, that each wide container contains at least one wide or small rectangle).

There are at most $1/\delta^2$ different types of long containers in C^w_L since all long containers have the same width and we rounded their heights to at most $1/\delta^2$ values. Each height is either a combination of the rounded heights of the long rectangles or a multiple of δ^4 (cf. Lemma 24). There are at most $(1/\delta^2 + 1)^{1/\delta}$ possibilities for the combinations of the rounded heights, since we have to choose at most $1/\delta$ values out of $1/\delta^2$ different heights. The additional 1 represents a dummy rectangle to choose less than $1/\delta$ values. We guess the heights of the long containers by choosing $1/\delta^2$ values out of $1/\delta^4 + (1/\delta^2 + 1)^{1/\delta}$. In a next step we guess the number n^ℓ_i of long containers of the ℓth height by choosing $1/\delta^2$ values out of n.

This is also done for the long and wide containers that are packed in the bins of Type 2. In total, we have to guess $2 \cdot (2/\delta^6 + 1/\delta^2) = 4/\delta^6 + 2/\delta^2$ values out of n and $2/\delta^2$ values out of $1/\delta^4 + (1/\delta^2)^{1/\delta}$. Note that each wide container has a width of at least $\delta^2/2$, no matter if it is packed in a bin of Type 1 or in a bin of Type 2. Furthermore, we can assume that $((3/2+22\delta) \cdot \mathrm{OPT} + 53) \leq n$, since otherwise we can pack each rectangle in a separate bin. Hence, the total height of the wide containers is bounded by $2/\delta^2 \cdot k \leq 2/\delta^2 \cdot ((3/2+22\delta) \cdot \mathrm{OPT} + 53) \leq 2/\delta^2 \cdot n$. The same holds for the total width of all long containers.

3.2 Packing the Rectangles

We assigned the rectangles to their corresponding sets. It remains to pack the small, wide and long rectangles into the containers and to pack the containers and the big rectangles into the bins.

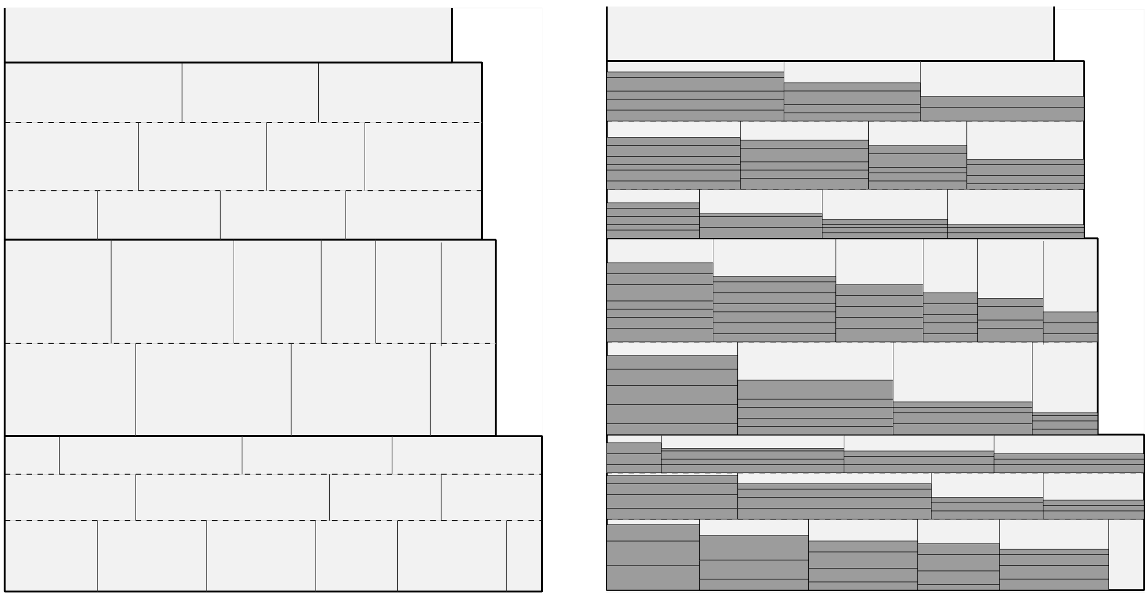
3.2.1 Packing Wide and Long Rectangles into the Containers

For packing the wide and long rectangles into the containers we use four similar linear programs. We explain the next steps for packing the wide rectangles into the wide containers, the packing for the long rectangles is analogous. A similar linear program formulation can be found in the $\mathcal{AFPTAS}$ by Kenyon & Rémila [19]. First, we focus on the wide containers of C^w_W that are packed in the bins of Type 1. Remember that all wide rectangles of W^w fit fractionally into the wide containers of C^w_W. There are at most $t := 2/\delta^2$ different widths of wide containers. We pack all containers of the same width $\ell \cdot \delta^2/2$, for $\ell \in \{1, \ldots, t\}$, on top of each other and treat them as one target region T_ℓ of height $h(T_\ell) = h(C^w_{W_\ell})$ and width $w(T_\ell) = \ell \cdot \delta^2/2$. The linear program will divide the target regions into slots of a certain width in which we will pack wide rectangles of the same width. Therefore, let $m := 2/\delta^2 - 2/\delta + 1$ be the number of different slots that have a width of $(i-1) \cdot \delta^2/2 + \delta$, for $i \in \{1, \ldots, m\}$. For each target region T_ℓ we define a set of configurations $\mathfrak{C}^{(\ell)}_j$. A configuration in $\mathfrak{C}^{(\ell)}_j$ consists of a set of at most $1/\delta$ slots that have a total width of at most $w(T_\ell) = \ell \cdot \delta^2/2$. The total number of possibilities to select at most $1/\delta$ slots out of m different types is $(m+1)^{1/\delta}$. Therefore, the total number of configurations $q^{(\ell)}$ for target region T_ℓ is bounded by the number $(m+1)^{1/\delta}$. The value $a(i, \mathfrak{C}^{(\ell)}_j)$ gives the number of slots of width $(i-1) \cdot \delta^2/2 + \delta$ in configuration $\mathfrak{C}^{(\ell)}_j$.

We solve the following feasibility linear program, where the variable $x_j^{(\ell)}$ describes the height of configuration $\mathfrak{C}_j^{(\ell)}$.

$$\sum_{j=1}^{q^{(\ell)}} x_j^{(\ell)} = h(T_\ell) \qquad \ell \in \{1,\dots,t\}$$
$$\sum_{\ell=1}^{t}\sum_{j=1}^{q^{(\ell)}} a(i,\mathfrak{C}_j^{(\ell)}) \cdot x_j^{(\ell)} \geq h(W_i^w) \qquad i \in \{1,\dots,m\}$$
$$x_j^{(\ell)} \geq 0 \qquad j \in \{1,\dots,q^\ell\}, \ell \in \{1,\dots,t\}$$

The first t constraints ensure that the total heights of the configurations do not exceed the total height of the target regions. We pack the wide rectangles of width $(i-1)\cdot\delta^2/2+\delta$ into the slots of the same width. To this end, the following m constraints ensure that the total height of the slots is large enough to occupy the wide rectangles (cf. Figure 30). For the right guess of the values above, this linear program computes a feasible solution. The corresponding matrix of the linear program has $t+m \leq 4/\delta^2$ rows and $q := \sum_{\ell=1}^{t} q^{(\ell)} \leq t\cdot(m+1)^{1/\delta} \leq (2/\delta^2)^{2/\delta+1}$ columns. Each entry of the matrix is bounded by $1/\delta$. Since the total height of all containers is bounded by $2/\delta^2 \cdot n$ the entries on the right side are bounded by $2/\delta^2 \cdot n$. It follows that the encoding length of the input is bounded by $L := (t+m)\cdot(q+1)\cdot\log(2/\delta^2\cdot n) \leq 4/\delta^2 \cdot ((2/\delta^2)^{2/\delta+1}+1)\cdot\log(2/\delta^2\cdot n) \leq 4\cdot(2/\delta^2)^{2/\delta+2}\cdot\log(2/\delta^2\cdot n)$. We can solve this linear program with a result of Vaidya [22], that computes a feasible basic solution in time $\mathcal{O}((((t+m)+q)q^2+((t+m)+q)^{3/2}q)L) = \mathcal{O}(q^3\cdot L) = \mathcal{O}((2/\delta^2)^{6/\delta+3}\cdot(2/\delta^2)^{2/\delta+2}\cdot\log(2/\delta^2\cdot n)) = \mathcal{O}((2/\delta^2)^{8/\delta+5}\cdot\log(2/\delta^2\cdot n))$



(a) The slots in the configurations computed by the linear program

(b) Packing the rectangles into the slots

Figure 30: Packing the wide rectangles into the containers

The rank of the matrix is bounded by the number of constraints which is at most $m+t = 2/\delta^2 - 2/\delta + 1 + 2/\delta^2 < 4/\delta^2$. It follows that the feasible basic solution contains less than

$4/\delta^2$ non-zero variables x_j^ℓ and thus we have less than $4/\delta^2$ configurations. We pack the wide rectangles of $W^w_{2/\delta}, \ldots, W^w_{2/\delta^2}$ greedily into the configurations, by packing them on top of each other into a slot of the same width until the last rectangle exceeds the height of the configuration. Since the rectangles fit fractionally into the slots, there is no rectangle unpacked. Afterwards, we remove the uppermost rectangles that exceed the height of the configuration. Therefore we pack the removed rectangles of one configuration next to each other into an additional bin. Thus, we need a total height of $\delta^4 \cdot 4/\delta^2 = 4\delta^2$ to pack all rectangles into one additional bin.

We sort the slots in each configuration by non-increasing packing heights, i.e. the leftmost slot occupies rectangles of the largest total height and the rightmost slot occupies rectangles of the smallest total height. The free space on the right of the configurations is separated into rectangular regions in order to pack small rectangles into it. This is done by drawing a horizontal line on the topmost rectangle in each slot. We have at most $1/\delta$ slots in each configuration and hence at most $1/\delta + 1$ different rectangular regions. In total, there are less than $(4/\delta^2) \cdot (1/\delta + 1) = 4/\delta^3 + 4/\delta^2 \leq 5/\delta^3$ different rectangular regions (cf. Figure 31).

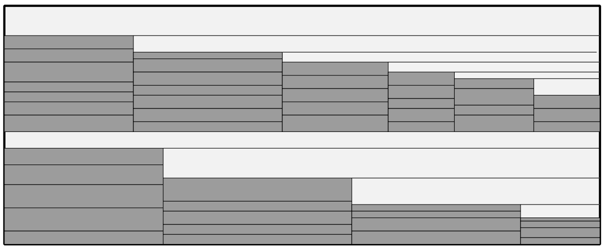


Figure 31: Rectangular regions for the small rectangles

We use the same linear program for the wide rectangles that are packed in wide containers in the bins of Type 2. The difference is that there are only $t = 1/\delta^2$ different target regions and that there are only $m = 1/\delta^2$ different widths of the rectangles and slots. Hence, we have only $m + t = 2/\delta^2$ constraints and therefore only $2/\delta^2$ different configurations. We pack the wide rectangles of $W^h_1, \ldots, W^h_{2/\delta^2}$ greedily into the configurations and we remove again the topmost rectangles. The removed rectangles are packed on top of the wide rectangles in the additional bin above. They need an additional space of height $\delta^4 \cdot 2/\delta^2 = 2\delta^2$. This results into a total packing height of at most $4\delta^2 + 2\delta^2 = 6\delta^2$. On the right side of these configurations we have at most $2/\delta^2 \cdot (1/\delta + 1) \leq 3/\delta^3$ rectangular regions.

We do the same steps for packing the long rectangles into the long containers by packing some remaining long rectangles into a second additional bin. Consequently, we have at most $2 \cdot (5/\delta^3 + 3/\delta^3) = 2 \cdot 8/\delta^3$ free rectangular regions for small rectangles. In the version that allows rotations we only use one additional bin for occupying the wide and the rotated long rectangles.

Solving the Linear Programs Approximately We give now a short description how to solve the linear programs approximately in order to reduce the running time. However, for the sake of readability we assume in the following sections that we have solved the linear programs exactly, as mentioned above. We solve each linear program approximately with an algorithm for the max-min resource sharing problem [11, 14] as explained in [3]. For a precision of δ^4 the algorithm stops after $\mathcal{O}(m(1/\delta^8 + \ln m)) = \mathcal{O}(1/\delta^{10})$ iterations. In each iteration a block problem has to be solved approximately with precision $\delta^4/6$. In our case the block problem consists of t knapsack problems with m unbounded variables. The t knapsack problems can be solved in time $\mathcal{O}(t \cdot (m \log(1/\delta^4) + 1/\delta^{16})) = \mathcal{O}(1/\delta^{18})$ [20]. The total running time is therefore $\mathcal{O}(1/\delta^{28})$. We obtain variables $\overline{x}_j^{(\ell)}$ that satisfies

$$\begin{aligned}
\sum_{\ell=1}^{t}\sum_{j=1}^{q^{(\ell)}} a(i,\mathfrak{C}_j^{(\ell)}) \cdot \overline{x}_j^{(\ell)} &\geq (1-\delta^4)h(W_i^w) && i \in \{1,\dots,m\}\\
\sum_{j=1}^{q^{(\ell)}} \overline{x}_j^{(\ell)} &= h(T_\ell) && \ell \in \{1,\dots,t\}\\
\overline{x}_j^{(\ell)} &\geq 0 && j \in \{1,\dots,q^\ell\}, \ell \in \{1,\dots,t\}.
\end{aligned}$$

The number of configurations is bounded by the number of iterations multiplied with t by $\mathcal{O}(t \cdot 1/\delta^{10}) = \mathcal{O}(1/\delta^{12})$. We reduce the number of configurations to $t+m$ by solving $\mathcal{O}(1/\delta^{12})$ systems of $t+m$ linear equalities with $t+m+1$ variables in time $\mathcal{O}((t+m)^3 \cdot 1/\delta^{10}) = \mathcal{O}(1/\delta^{16})$ as explained in [14].

In order to secure the covering constraints we extend each configuration by setting $x_j^{(\ell)} := (1+2\delta^4)\overline{x}_j^{(\ell)}$. Since $\delta^4 \leq 1/2$ we have for each $i \in \{1,\dots,m\}$

$$\sum_{\ell=1}^{t}\sum_{j=1}^{q^{(\ell)}} a(i,\mathfrak{C}_j^{(\ell)}) \cdot x_j^{(\ell)} \geq (1+2\delta^4)(1-\delta^4)h(W_i^w) = (1+\delta^4-2\delta^8)h(W_i^w) \geq h(W_i^w).$$

The heights of the configurations are extended for each target region T_ℓ to $\sum_{j=1}^{q^{(\ell)}} x_j^{(\ell)} = (1+2\delta^4)h(T_\ell)$.

We approximately solve also the linear program for packing the wide rectangles into the bins of Type 2 and pack all wide rectangles as described above. Afterwards we have to remove the rectangles in the uppermost strips of each target region. There are $2/\delta^2 + 1/\delta^2 = 3/\delta^2$ target regions for the wide rectangles that are packed in bins of both types. We remove strips of wide rectangles of the total height at most $2\delta^4 h(T_\ell) + \delta^4$ from each target region. The total height of all target regions is bounded by $2/\delta^2 \cdot ((3/2 + 22\delta) \cdot \mathrm{OPT} + 53)$ (each target region has a width of at least $2/\delta^2$). Therefore, we remove strips of wide rectangles of the total height

$$\begin{aligned}
2\delta^4 \cdot 2/\delta^2 \cdot ((3/2+22\delta) \cdot \mathrm{OPT} + 53) + \delta^4 \cdot 3/\delta^2 &=\\
4\delta^2 \cdot ((3/2+22\delta) \cdot \mathrm{OPT} + 53) + 3\delta^2 &\leq\\
(6\delta^2 + 88\delta^3)\mathrm{OPT} + 215\delta^2 &\leq\\
(6\delta^2 + 88\delta^2/48)\mathrm{OPT} + 215/48^2 &\leq\\
8\delta^2\mathrm{OPT} + 1&
\end{aligned}$$

We pack these strips on top of each other and cut the packing on each integral height in order to pack the rectangles in $\lceil 8\delta^2\mathrm{OPT}+1 \rceil$ additional bins. We remove the rectangles that are split by these cutting lines and pack them separately. We are able to pack rectangles of $1/\delta^4$ cutting lines on top of each other into one bin. Therefore, we need $\lceil \delta^4 \cdot (8\delta^2\mathrm{OPT}+1) \rceil \leq \lceil \delta^2\mathrm{OPT}+1 \rceil$ additional bins. In total we need less than $9\delta^2\mathrm{OPT}+4$ additional bins. The same holds for packing the long rectangles into the target regions.

3.2.2 Packing Small Rectangles into Rectangular Regions

We pack the small rectangles into the rectangular regions defined above. Remember that they fit fractionally into these regions. We have at most $2 \cdot 8/\delta^3$ different rectangular regions for the small rectangles. We use Next Fit Decreasing Height by Coffman et al. [7] to pack them into these regions. Since these rectangles are small we are able to cover almost the whole region with small rectangles.

We give a short description of this algorithm for the sake of completeness. Next Fit Decreasing Height sorts the rectangles according to non-increasing heights. In this order the algorithm packs the rectangles left-justified on a level, until there is insufficient space at the right to accommodate the next rectangle. This level is closed and the algorithm packs no further rectangle on it. If this is the first level, the algorithm packs the level on the ground of the target region, in any other case, we place this level on top of the first rectangle of the previous level. Then the algorithm proceeds packing on the next level, until it runs out of rectangles, or the next level does not fit into the target region.

We obtain the following Theorem by Coffman et al. [7]:

Lemma 26. *Let A be a rectangular region of width w_A and height h_A. We are able to pack small rectangles into A with a total area of at least $w_A \cdot h_A - (w_A + h_A) \cdot \delta^4$.*

Proof. Let t be the number of levels $L_1, \dots, L_t$ that are packed with Next Fit Decreasing Height into region A. We suppose that we have enough small rectangles and the algorithm stops, because the next level does not fit into A. Let r_i be the first rectangle on the level L_i and let $r_{i'}$ be the last rectangle on the level L_i. The height of the level L_i is the height h_i of the first rectangle on this level. Additionally, let $h_{t+1} = h_A - \sum_{i=1}^{t} h_i$ be the free space on top of the level L_t. Consequently, we have $\sum_{i=1}^{t+1} h_i = h_A$. It holds $h_{i'} \geq h_{i+1}$ since the rectangles are sorted according to non-increasing heights. Furthermore, let $w(L_i)$ be the total width of the rectangles on level L_i. We have $w(L_i) > w_A - \delta^4$ since the next small rectangle of width at most δ^4 does not fit on this level. The total area of the rectangles on level L_i is hence $a(L_i) \geq h_{i'} \cdot w(L_i) \geq h_{i+1} \cdot w(L_i) > h_{i+1} \cdot (w_A - \delta^4)$. Thus, the total area of the packed rectangles is at least

$$\sum_{i=1}^{t} a(L_i) > \sum_{i=1}^{t} h_{i+1} \cdot (w_A - \delta^4) = (h_A - h_1) \cdot (w_A - \delta^4) >$$
$$(h_A - \delta^4) \cdot (w_A - \delta^4) > w_A \cdot h_A - (w_A + h_A) \cdot \delta^4.$$

□

The total height of the wide containers in the bins of Type 1 and Type 2 is bounded by $2/\delta^2 \cdot ((3/2 + 22\delta) \cdot \mathrm{OPT} + 53)$ (each wide container has a width of at least $\delta^2/2$). Thus, the sum of the heights of the rectangular regions on the right side of the configurations with the wide rectangles is bounded by $2/\delta^2 \cdot ((3/2 + 22\delta) \cdot \mathrm{OPT} + 53)$. The width of each wide container is at most 1 and hence the sum of the widths of the at most $8/\delta^3$ rectangular regions is bounded by $8/\delta^3$. Therefore, in all target regions A for the wide rectangles there is only a free total area of at most

$$\begin{aligned}
\sum_A (w_A + h_A) \cdot \delta^4 = \delta^4 \cdot (\sum_A w_A + \sum_A h_A)& \\
&\leq \delta^4 \cdot (8/\delta^3 \cdot 1 + 2/\delta^2 \cdot ((3/2 + 22\delta) \cdot \mathrm{OPT} + 53)) \\
&= 8\delta + 2\delta^2 \cdot ((3/2 + 22\delta) \cdot \mathrm{OPT} + 53) \\
&= 3\delta^2 \cdot \mathrm{OPT} + 44\delta^3 \cdot \mathrm{OPT} + 8\delta + 106\delta^2 \\
&\leq 3\delta^2 \cdot \mathrm{OPT} + 44\delta^2/48 \cdot \mathrm{OPT} + 8\delta + 106\delta/48 \\
&< 4\delta^2 \cdot \mathrm{OPT} + 11\delta
\end{aligned}$$

left. The same bound holds also for the target areas for the long rectangles and we obtain a total free area of at most $2 \cdot (4\delta^2 \cdot \mathrm{OPT} + 11\delta) \leq 8\delta^2 \cdot \mathrm{OPT} + 22\delta$. Since all small rectangles fit fractionally into the containers, it follows that the total area of the unpacked small rectangles is bounded by this value.

These small rectangles fit with Lemma 26 into $\delta\mathrm{OPT} + 1$ additional bins, since we are able to pack small rectangles of a total area at least $1 - (1+1) \cdot \delta^4 = 1 - 2\delta^4$ into one bin and we have

$$\begin{aligned}(1 - 2\delta^4) \cdot (\delta\mathrm{OPT} + 1) &= \delta\mathrm{OPT} - 2\delta^5\mathrm{OPT} + 1 - 2\delta^4 \\ &\geq 48\delta^2\mathrm{OPT} - 2\delta^5\mathrm{OPT} + 48\delta - 2\delta^4 \geq 8\delta^2\mathrm{OPT} + 22\delta.\end{aligned}$$

3.2.3 Cutting Out Containers

We treated all wide containers of the same width and all long containers of the same height as one target region. The total number of containers is bounded in Lemma 24 by $6/\delta^3 \cdot p_1 + 2 + 2/\delta^3 \cdot (k - p_1) + 2/\delta^2$ wide containers and $6/\delta^3 \cdot (k-p_1) + 2 + 2/\delta^3 \cdot p_1 + 2/\delta^2$ long containers. It is left to cut the containers out of the target regions. We cut hereby wide and small rectangles of height δ^4 or long and small rectangles of width δ^4. Hence, we are able to pack the cut rectangles of $1/\delta^4$ horizontal or vertical cut lines into one additional bin. Consequently, we need

$$\begin{aligned}
\lceil \delta^4 \cdot (6/\delta^3 \cdot p_1 + 2 + 2/\delta^3 \cdot (k - p_1) + 2/\delta^2) \rceil + \\
\lceil \delta^4 \cdot (6/\delta^3 \cdot (k - p_1) + 2 + 2/\delta^3 \cdot p_1 + 2/\delta^2) \rceil \leq \\
\delta^4 \cdot (8/\delta^3 \cdot k + 4/\delta^2 + 4) + 2 \leq \\
\delta^4 \cdot (8/\delta^3 \cdot ((3/2 + 5\delta)\mathrm{OPT} + 37) + 4/\delta^2 + 4) + 2 = \\
8\delta \cdot ((3/2 + 5\delta)\mathrm{OPT} + 37) + 4\delta^2 + 4\delta^4 + 2 = \\
8\delta \cdot (3/2 + 5\delta)\mathrm{OPT} + 8\delta \cdot 37 + 4\delta^2 + 4\delta^4 + 2 = \\
(12\delta + 40\delta^2)\mathrm{OPT} + 296\delta + 4\delta^2 + 4\delta^4 + 2 \leq \\
(12\delta + 40\delta/48)\mathrm{OPT} + 296/48 + 4/48^2 + 4/48^4 + 2 \leq \\
13\delta\mathrm{OPT} + 7 + 2 = \\
13\delta\mathrm{OPT} + 9
\end{aligned}$$

additional bins.

Cutting Out Containers with Rotations The number of wide containers and long containers in the version that allows rotation is bounded in Lemma' 24 by $6/\delta^3 \cdot k' + 2$ and $2/\delta^3 \cdot k' + 2/\delta^2$, respectively. Therefore, we have $6/\delta^3 \cdot k' + 2 + 2/\delta^3 \cdot k' + 2/\delta^2 = 8/\delta^3 \cdot k' + 2/\delta^2 + 2$ cutting lines. We rotate the rectangles that are cut by the construction of the long containers. Thus, we have only horizontal cutting lines and thus cut rectangles of height δ^4. Consequently, we need $\lceil \delta^4 \cdot (8/\delta^3 \cdot k' + 2/\delta^2 + 2) \rceil \leq \delta^4 \cdot (8/\delta^3 \cdot k' + 2/\delta^2 + 2) + 1$ bins. We obtain

$$\begin{aligned}
\delta^4 \cdot (8/\delta^3 \cdot k' + 2/\delta^2 + 2) + 1 \leq \\
\delta^4 \cdot (8/\delta^3 \cdot ((3/2 + 5\delta)\mathrm{OPT} + 21) + 2/\delta^2 + 2) + 1 \leq \\
8\delta \cdot (3/2 + 5\delta)\mathrm{OPT} + 8\delta \cdot 21 + 2\delta^2 + 2\delta^4 + 1 \leq \\
13\delta\mathrm{OPT} + 168\delta + 2\delta^2 + 2\delta^4 + 1 \leq \\
13\delta\mathrm{OPT} + 168/48 + 2/48^2 + 2/48^4 + 1 \leq \\
13\delta\mathrm{OPT} + 4 + 1 = \\
13\delta\mathrm{OPT} + 5
\end{aligned}$$

additional bins.

3.2.4 Packing Big Rectangles and Containers

The last remaining step is to pack the big rectangles and the long and wide containers into the bins. Therefore, we use almost the same linear program as above. Again, we explain the following steps for the bins of Type 1. One configuration C_j, for $j \in \{1, \dots, q\}$, consists of a packing into one bin. There are at most $2/\delta^2 \cdot 1/\delta^2 = 2/\delta^4$ different types of big rectangles, $2/\delta^6$ different types of wide containers and $1/\delta^2$ different types of long container. In each bin/configuration there are at most $1/\delta^2$ big rectangles, $6/\delta^3$ wide containers and $2/\delta^3$ long containers (see description above Lemma 24). Therefore, there are at most $(2/\delta^4 + 1)^{1/\delta^2}$ possibilities to select at most $1/\delta^2$ big rectangles out of $2/\delta^4$ different types. The additional 1 represents a dummy rectangle and is needed for selecting less than k big rectangles. There are at most $(2/\delta^6 + 1)^{6/\delta^3}$ possibilities to select at most $6/\delta^3$ wide containers out of $2/\delta^6$ different types and $(1/\delta^2 + 1)^{2/\delta^3}$ possibilities to select at most $2/\delta^3$ long containers out of $1/\delta^2$ different types. All together, the number q of different configurations is therefore bounded by

$$
\begin{aligned}
q &\le (2/\delta^4 + 1)^{1/\delta^2} \cdot (2/\delta^6 + 1)^{6/\delta^3} \cdot (1/\delta^2 + 1)^{2/\delta^3} \\
&< (4/\delta^4)^{1/\delta^2} \cdot (4/\delta^6)^{6/\delta^3} \cdot (2/\delta^2)^{2/\delta^3} \\
&= (1/\delta)^{4/\delta^2} \cdot (1/\delta)^{36/\delta^3} \cdot (1/\delta)^{4/\delta^3} \cdot 2^{2/\delta^2} \cdot 2^{12/\delta^3} \cdot 2^{2/\delta^3} \\
&\le (1/\delta)^{41/\delta^3} \cdot 2^{15/\delta^3}.
\end{aligned}
$$

We have to verify, if a candidate for a configuration fits into a bin. Each wide and long container and each big rectangle has a width of a multiple of $\delta^2/2$. Therefore, we are able to pack them with its x-coordinate on a multiple of $\delta^2/2$. For each candidate we guess the x-coordinates of all containers and big rectangles by choosing $1/\delta^2 + 6/\delta^3 + 2/\delta^3 \le 9/\delta^3$ values out of $2/\delta^2$. Consequently, we have for each multiple of $\delta^2/2$ one set of big rectangles and containers that starts on the corresponding x-coordinate or intersect this x-coordinate completely. It remains to find an order of these containers and big rectangles to find a packing. Since there are at most $1/\delta^4$ objects in each set, there are at most $1/\delta^4!$ possible permutations. In total we have to try $2/\delta^2 \cdot 1/\delta^4! \le 2/\delta^2 \cdot (1/\delta^4)^{1/\delta^4}$ permutations to find a packing of this configuration. If we do not find a packing at all, then there exists no packing of this configuration and we delete it.

Afterwards, we select the configurations that have to be packed into the bins. We need for each configuration one bin. To select these configurations we employ an integer linear program. Therefore, denote with $b(i, j, C_k)$ the number of big rectangles in the set $B^w_{i,j}$ in configuration C_k. With $w(i, j, C_k)$, we denote the number of wide containers of width $i\delta^2/2$ and height $j\delta^4$ and with $\ell(i, C_k)$ we denote the number of long containers of the ith height in configuration C_k. The total number of big rectangles in the set $B^w_{i,j}$ is denoted by $n^b_{i,j}$, the total number of wide containers of the width $i\delta^2/2$ and of the height $j\delta^4$ is denoted by $n^w_{i,j}$ and the number of long containers of the ith height by n^ℓ_i.

The integer linear program is defined as follows:

$$
\begin{aligned}
&\min \sum_{k=1}^{q} x_k \\
s.t. &\sum_{k=1}^{q} b(i,j,C_k)\cdot x_k \geq n^b_{i,j} && i \in \{2/\delta,\dots,2/\delta^2\}, j \in \{1,\dots,1/\delta^2\}\\
&\sum_{k=1}^{q} w(i,j,C_k)\cdot x_k \geq n^w_{i,j} && i \in \{2/\delta,\dots,2/\delta^2\}, j \in \{1,\dots,1/\delta^4\}\\
&\sum_{k=1}^{q} \ell(i,C_k)\cdot x_k \geq n^\ell_{i} && i \in \{1,\dots,1/\delta^2\}\\
&x_k \in \mathbb{N} && k \in \{1,\dots,q\}
\end{aligned}
$$

This integer linear program can be solved with the algorithm of Kannan [18] in time $q^{\mathcal{O}(q)}\cdot s$, while s is the input size. We have $(2/\delta^2 - 2/\delta)\cdot 1/\delta^2 + (2/\delta^2 - 2/\delta)\cdot 1/\delta^4 + 1/\delta^2 \leq 3/\delta^6$ constraints. Each coefficient in the matrix is bounded by $6/\delta^3$ and the values $n^b_{i,j}, n^w_{i,j}$ and n^ℓ_i are bounded by n. It follows, that $s \leq (q+1)\cdot 3/\delta^6 \cdot \log(n)$.

Thus, the the total running time, including the construction of the configurations, is bounded by $\mathcal{O}(\log(n)\cdot q^{\mathcal{O}(q)})$. We can improve the running-time with a result of Eisenbrand & Shmonin [9]:

Theorem 7. *Let $X \subset \mathbb{Z}^d$ be a finite set of integer vectors and let $b \in \{\sum_{i=1}^{t}\lambda_i x_i | t \geq 0; x_1,\dots,x_t \in X; \lambda_1,\dots\lambda_t \in \mathbb{Z}_{\geq 0}\}$. Then there exists a subset $\tilde{X} \subseteq X$ such that $b \in \{\sum_{i=1}^{t}\lambda_i x_i | t \geq 0; x_1,\dots,x_t \in \tilde{X}; \lambda_1,\dots\lambda_t \in \mathbb{Z}_{\geq 0}\}$ and $|\tilde{X}| \leq 2d\log(4dM)$ with $M = \max_{x\in X}\|x\|_\infty$.*

In our case, the set X belongs to the configurations. We have at most $3/\delta^6$ constraints, thus $d \leq 3/\delta^6$. The coefficients of the matrix are bounded by $M = 6/\delta^3$. Theorem 7 states that there are at most $q' := 2d\log(4dM) \leq 2\cdot 3/\delta^6\log(4\cdot 3/\delta^6\cdot 6/\delta^3) \leq 6/\delta^6\log(62/\delta^9)$ non-zero variables in any solution b of our integer linear program. We enumerate all configurations of cardinality at most q' and have at most $(q+1)^{q'}$ possibilities. For each set of at most q' configurations we solve the reduced integer linear program with the algorithm of Kannan [18] in time $q'^{\mathcal{O}(q')}\cdot s'$ while s' is the input size of the reduced integer linear program. We bound s' by $(q'+1)\cdot d\cdot\log(n)$. Hence, the total running time is bounded by $\mathcal{O}((q+1)^{q'}\cdot\log(n)\cdot q'^{\mathcal{O}(q')})$.

The same integer linear program is solved for the bins of Type 2. Since we know that there is a packing into $(3/2+24\delta)\cdot\mathrm{OPT}+53$, these integer linear programs compute for the right guess of all above described values a solution with at most $(3/2+24\delta)\cdot\mathrm{OPT}+53$ bins.

3.3 Résumé of the Algorithm

A compressed version of our algorithm is given in Algorithm 1.

The running time of the steps are given as follows. The binary search takes $\mathcal{O}(\log n)$ time. To find δ, we have to compute $2/\varepsilon'$ sets and check whether their value is at most $\varepsilon'\cdot\mathrm{OPT}$. This takes time $\mathcal{O}(n\cdot 2/\varepsilon') = \mathcal{O}(n/\varepsilon)$.

We pack the $3\varepsilon'\mathrm{OPT}+2$ sets with the algorithm of Steinberg that has a running time of $\mathcal{O}((n\log^2 n)/\log\log n)$. Since $\mathrm{OPT} \leq n$ and $\varepsilon' < 1$, we obtain a total running time for this step of $\mathcal{O}((n^2\log^2 n)/\log\log n)$. The value of δ is at least $\delta \geq \varepsilon'^{4\cdot 2/\varepsilon'}$. For the structure of the sets for the big rectangles we have to guess $4/\delta^2 + 4/\delta^4$ values out of n. We obtain the structure of the sets of the wide and long rectangles by guessing at most $6/\delta^3$ values out of n.

Algorithm 1 Algorithm for Two-Dimensional Bin Packing

1: set $\varepsilon' := \min\{\varepsilon/39, 1/48\}$
2: **Find** $\mathrm{OPT}' \leq \mathrm{OPT}$ with binary search so that algorithm computes feasible solution **for each guess do**
3: Compute δ and pack medium rectangles with Steinberg's Theorem 5
4: **Find** structure of the set of big, long and wide rectangles and of the set of wide and long containers **for each guess do**
5: Solve flow network with the algorithm of Dinic [8]
6: Greedy assignment of long and wide rectangles into groups
7: Pack the long and wide rectangles into containers with linear programs that are solved by the algorithm of Vaidya [22]
8: Pack the small rectangles with Next Fit Decreasing Height by Coffman et al. [7]
9: Pack containers and big rectangles with integer linear programs that are solved by the algorithm of Kannan [18]

We compute the structure of the wide and long containers by guessing $4/\delta^6 + 2/\delta^2$ values out of n and $2/\delta^2$ values out of $1/\delta^4 + (1/\delta^2)^{1/\delta}$. In total we have to choose less than $5/\delta^6$ values out of n and $2/\delta^2$ values out of $1/\delta^4 + (1/\delta^2)^{1/\delta}$. This takes time $\mathcal{O}(n^{5/\delta^6} \cdot (1/\delta)^{2/\delta})$. To solve the flow network, we need time $\mathcal{O}(n^3/\delta^2 + n^2/\delta^6 + n/\delta^{10} + 1/\delta^{14})$. The assignment of the long and wide rectangles into the groups is done in linear time. We solve the four linear programs to pack the wide and long rectangles into the containers in time $\mathcal{O}((2/\delta^2)^{8/\delta+5} \cdot \log(2/\delta^2 \cdot n))$. The packing of the small rectangles and to cut out the containers afterwards is done in less than $\mathcal{O}(n \log n/\delta^3)$ time. The integer linear programs are solved in time $\mathcal{O}((q+1)^{q'} \cdot \log(n) \cdot q'^{\mathcal{O}(q')})$, with $q \leq (1/\delta)^{41/\delta^3} \cdot 2^{15/\delta^3}$ and $q' \leq 6/\delta^6 \log(62/\delta^9)$.

To conclude, the running time is bounded by $\mathcal{O}(n^{f(1/\varepsilon)} \cdot g(1/\varepsilon))$ for some functions f and g. We obtain the following result for the two-dimensional bin packing problem with and without rotations:

Theorem 1. *For any $\varepsilon > 0$, there is an approximation algorithm A which produces a packing of a list I of n rectangles in $A(I)$ bins such that*

$$A(I) \leq (3/2 + \varepsilon) \cdot \mathrm{OPT}(I) + 69.$$

The running time of A is polynomial in n.

Proof. The integer linear programs packs the big rectangles and the containers in at most $(3/2 + 22\delta) \cdot \mathrm{OPT} + 53$ bins. The medium rectangles are packed into $3\varepsilon'\mathrm{OPT} + 2$ bins. To transform the wide and long rectangles we need 2 additional bins. We pack the wide and long rectangles with the linear programs into the target regions. Therefore, we need also 2 additional bins. The small rectangles are packed into the target regions and into $\delta\mathrm{OPT} + 1$ additional bins. Afterwards, we cut the containers out of the target regions and need $13\delta\mathrm{OPT} + 9$ additional bins. It follows that we need

$$(3/2 + 22\delta) \cdot \mathrm{OPT} + 53 + 3\varepsilon'\mathrm{OPT} + 2 + 2 + 2 + \delta\mathrm{OPT} + 1 + 13\delta\mathrm{OPT} + 9 \leq (3/2 + 39\varepsilon') \cdot \mathrm{OPT} + 69$$

bins. Since $\varepsilon' \leq \varepsilon/39$ we obtain $(3/2 + \varepsilon)\mathrm{OPT} + 69$ bins in total. □

Résumé of the Algorithm with Rotations In the version that allows rotation, the integer linear program packs the rectangles into $k' \leq (3/2 + 22\delta) \cdot \mathrm{OPT} + 30$ bins. The medium rectangles are packed into $3\varepsilon' \cdot \mathrm{OPT} + 1$ additional bins. To assign the long and wide rectangles to the groups and to pack them into the target regions we need in total 2 additional bins. To pack the small rectangles we need $\delta\mathrm{OPT} + 1$ additional bins and to cut the containers out of the target regions we need $13\delta\mathrm{OPT} + 5$ additional bins. Consequently, the total number of used bins is at most

$$(3/2 + 22\delta) \cdot \mathrm{OPT} + 30 + 3\varepsilon' \cdot \mathrm{OPT} + 1 + 2 + \delta\mathrm{OPT} + 1 + 13\delta\mathrm{OPT} + 5 \leq \\ (3/2 + 39\varepsilon') \cdot \mathrm{OPT} + 39 \leq (3/2 + \varepsilon) \cdot \mathrm{OPT} + 39$$

bins. We obtain the following result:

Theorem' 1. *For any $\varepsilon > 0$, there is an approximation algorithm A which produces a packing of a list I of n rectangles that are allowed to be rotated in $A(I)$ bins such that*

$$A(I) \leq (3/2 + \varepsilon) \cdot \mathrm{OPT}(I) + 39.$$

The running time of A is polynomial in n.

4 Conclusion

We presented a technique that allows us to modify any solution of the two-dimensional bin packing problem into a solution that consists of a simpler structure. This enables our algorithm to compute a solution into $(3/2 + \varepsilon) \cdot \mathrm{OPT} + 69$ bins and an improved solution of $(3/2 + \varepsilon) \cdot \mathrm{OPT} + 39$ in the version that allows rotation for any fixed $\varepsilon > 0$ and any instance that fits optimally into OPT bins. The current lower bound is given by Chlebík & Chlebíková [5] with values $1 + 1/3792$ and $1 + 1/2196$ for the version with and without rotations, respectively. An open question is to close the gap between the current lower bounds and our presented asymptotic approximation ratios. Therefore, it is of interest to find an approximation algorithm with an asymptotic approximation ratio of $4/3$, if there is any. Maybe there is a way to adopt our techniques by modifying an optimal solution so that the rectangles are rounded up. However, there would be only one additional bin for each sequence of three bins, instead of one additional bin for each sequence of two bins. Therefore. it would be necessary to do a much more sensitive and complex case analysis.

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